BOUNDING ZEROS OF \( H^2 \) FUNCTIONS VIA CONCENTRATIONS

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**Abstract.** It is well-known that the zeros \( \{z_j\} \) of a function in the classical Hardy space \( H^2 \) satisfy \( \sum 1 - |z_j| < \infty \); however, this sum can be arbitrarily large. We shall bound this sum by a constant that depends on the concentration of the function, a concept introduced by Beauzamy and Enflo.

1. INTRODUCTION

Consider a function \( f : D \to \mathbb{C} \) in the classical Hardy space \( H^2(D) \) where \( D \) is the open unit disk in the complex plane \( \mathbb{C} \). It is well-known that the zeros \( \{z_j\} \) of \( f \) satisfy \( \sum 1 - |z_j| < \infty \). However, this sum can be arbitrarily large as seen by considering an appropriate Blaschke product.

Fix \( 1 \leq p \leq 2 \) and consider the subset \( \mathcal{A}_p \) of \( H^2 \) where

\[
\mathcal{A}_p = \{ f \in H^p(D) : f(z) = \sum_{j \geq 0} a_j z^j \text{ and } \sum_{j \geq 0} |a_j|^p < \infty \}.
\]

Of course, \( \mathcal{A}_2 \) is just \( H^2 \) and \( \mathcal{A}_1 \) is the usual algebra \( \mathcal{A}_+(T) \). For a function in \( \mathcal{A}_p \) with zeros \( \{z_j\} \), we shall bound \( \sum 1 - |z_j| \) by a constant that depends on the concentration of the function, a notation introduced by Beauzamy and Enflo [BE].

Now to recall basic facts and fix notation. For

\[
f(z) = \sum_{j \geq 0} a_j z^j \in \mathcal{A}_p,
\]

the \( l_p \)-norm of \( f \) is

\[
\|f\|_p = \left[ \sum_{j \geq 0} |a_j|^p \right]^{1/p},
\]

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and the $k$-th partial sum $s_k(f)$ of $f$ is

$$[s_k(f)](z) = \sum_{j=0}^{k} a_j z^j.$$  

The function $f \in A_p$ has concentration $d$ at degree $k$ (measured in $\ell_p$-norm) if

$$d \parallel f \parallel_p \leq \parallel s_k(f) \parallel_p$$

where $0 < d \leq 1$ and $k$ is a non-negative integer. The largest $d$ which satisfies the above condition is called the concentration factor.

An $H^2$ function $f$ has a canonical factorization $f(z) = F(z)B(z)S(z)$ into an outer function $F$, a Blaschke product $B$, and a singular part $S$. The measure $M(f)$ of $f$ is

$$M(f) = \exp\left[\int_0^{2\pi} \log |f(e^{i\theta})| \frac{d\theta}{2\pi}\right].$$

Recall $M(f) = |F(0)| \leq \| F \|_2 = \| f \|_2$ and since $1 \leq p \leq 2$ furthermore $\| f \|_2 \leq \| f \|_p$.

For $0 \leq \lambda \leq 1$, let $f_\lambda : D \to \mathbb{C}$ be the function $f_\lambda(z) = f(\lambda z)$. Throughout this paper, we will assume that $f(0) \neq 0$ and enumerate the zero set $\{z_j\}$ of $f$ so that

$$0 < |z_1| \leq |z_2| \leq \ldots < 1.$$  

All notation and terminology, not otherwise explained, are as in [G] or [B2].

As motivation we examine a function $f \in A_p$ with concentration $d$ at degree $k = 0$. For such a function we have that

$$d \parallel f \parallel_p \leq \parallel s_0(f) \parallel_p = |f(0)| = |F(0)| S(0) \prod_{j>0} |z_j|.$$  

We know that the singular part $S$ of an $H^2$ function satisfies $0 \leq S(0) \leq 1$. Thus

$$d \leq \prod_{j>0} |z_j|,$$

and since $1 - x \leq -\log x$ for $0 < x \leq 1$, we see that

$$\sum_{j>0} 1 - |z_j| \leq -\log d.$$  

The first bound is essentially best possible. Consider the family of $H^2$ functions $\{f_\varepsilon(z) = z - \varepsilon : 0 < \varepsilon < 1\}$. For $1 \leq p \leq 2$, the function $f_\varepsilon$ has concentration $d = \frac{\varepsilon}{1+\varepsilon}$ at degree $0$. Thus

$$d \leq \prod_{j>0} |z_j| = \varepsilon \leq d(1 + \varepsilon).$$

We now extend these ideas for concentration at an arbitrary degree $k$. 

2. ZEROS OF FUNCTIONS IN $A_p$

Beauzamy and Chou [BC] showed that the zeros $\{z_j\}$ of a polynomial with concentration $d$ at degree $k$ (measured in $\ell_1$-norm) satisfy

$$\frac{d^{k+3}}{2 \cdot 9^{k+3} \cdot 3^{k^2}} \leq \prod_{j>k} |z_j|.$$ 

Actually, their product includes the zeros outside the unit disk as well. Theorem 2.1 extends this result to functions in $A_p$ and improves their constant. For our constant, we consider the function

$$\phi_{d,k}(r) \equiv \left[ \frac{(1-r)r^k d}{e^{\frac{1}{1-r}}} \right]^{\frac{1}{1+r}}$$

and let

$$\Phi_{d,k} \equiv \max_{0<r<1} \phi_{d,k}(r).$$

Recall that we enumerate a zero set $\{z_j\}$ so that $0 < |z_1| \leq |z_2| \leq \ldots < 1$. We now fix $1 \leq p \leq 2$.

**Theorem 2.1.** The zeros $\{z_j\}$ of a function in $A_p$ with concentration $d$ at degree $k$ (measured in $\ell_p$-norm) satisfy

$$d \Phi_{d,k} \leq \prod_{j>k} |z_j| \quad \text{and thus} \quad \sum_{j>0} 1 - |z_j| \leq k - \log (d \Phi_{d,k}).$$

To compare with [BC], note that

$$\frac{d^{k+3}}{2 \cdot 9^{k+3} \cdot 3^{k^2}} < \frac{4d^2}{e^{4+1}} = d \phi_{d,k}(\frac{1}{2}) \leq d \Phi_{d,k}.$$

The proof uses the following lemma which extends a result of Beauzamy ([B1], [B2]).

**Lemma 2.2.** If a function $f$ in $A_p$ has concentration $d$ at degree $k$, then

$$\Phi_{d,k} \| f \|_p \leq M(f).$$

As noted earlier $M(f) \leq \| f \|_p$ since $1 \leq p \leq 2$.

**Proof.** Let $f(z) = \sum_{j \geq 0} a_j z^j$ be a function in $A_p$ with appropriate concentration. Normalize so that $\| f \|_p = 1$. Fix $0 < r < 1$ and consider $z_0 \in D$ with $|z_0| = r$ along with its Möbius function $w(z) = \frac{z + z_0}{1 + z \cdot z_0}$ defined on $D$. Jensen’s formula and a change of variables give that

$$\log |f(z_0)| \leq \int_0^{2\pi} \log |f(w(e^{i\theta}))| \frac{d\theta}{2\pi} = \int_0^{2\pi} \log |f(e^{i\theta})| \left| \frac{1 - r^2}{1 - \frac{z_0}{z} e^{i\theta}} \right| \frac{d\theta}{2\pi} \leq \frac{1 - r}{1 + r} \int_{\frac{1}{2}} \log |f(e^{i\theta})| \frac{d\theta}{2\pi} + \frac{1 + r}{1 - r} \int_{\frac{1}{2}} \log |f(e^{i\theta})| \frac{d\theta}{2\pi},$$
where
\[ N = \{ \theta : |f(e^{i\theta})| \leq 1 \} \quad \text{and} \quad P = \{ \theta : |f(e^{i\theta})| > 1 \}. \]

For bounding the last summand, note that
\[
\int_P \log |f(e^{i\theta})| \frac{d\theta}{2\pi} \leq \frac{1}{2} \int |f|^2 \frac{d\theta}{2\pi} = \frac{1}{2} \| f \|_2^2 \leq \frac{1}{2} \| f \|_p^2 = \frac{1}{2}.
\]

Choose \( z_0 \) such that \( |f(z_0)| = \max \{|f(z)| : |z| = r\} \). Since
\[
a_j = \int_0^{2\pi} \frac{f(re^{i\theta})}{r^j e^{i\theta}} \frac{d\theta}{2\pi} \quad \text{and so} \quad |a_j| \leq \frac{|f(z_0)|}{r^j},
\]

it follows that
\[
\| s_k(f) \|_p \leq \| s_k(f) \|_1 = \sum_{j=0}^k |a_j| \leq |f(z_0)| \sum_{j=0}^k \frac{1}{r^j} \leq |f(z_0)| \frac{r^{-k} \sum_{j=0}^\infty r^j}{(1 - r)r^k} = \frac{|f(z_0)|}{(1 - r)r^k}.
\]

Combining this with the concentration information on \( f \) we see that
\[
(1 - r)r^k d \leq |f(z_0)|.
\]

We now have that
\[
\log ((1 - r)r^k d) - \frac{1 + r}{2(1 - r)} \leq \frac{1 - r}{1 + r} \int_N \log |f(e^{i\theta})| \frac{d\theta}{2\pi},
\]

which gives that \( \phi_{d,k}(r) \leq M(f) \), as needed.

Keeping with the previous notation, we observe a property of the measure.

**Lemma 2.3.** If the \( \mathcal{H}^2 \) function \( f \) has a \( k \)-th zero of modulus \( \lambda \), then
\[
M(f_\lambda) = \lambda^k |F(0)| \prod_{j > k} |z_j|.
\]

**Proof.** Since \( f_\lambda \) is analytic on the closed unit disk \( \overline{D} \), Jensen’s formula provides that
\[
M(f_\lambda) = \exp \int_0^{2\pi} \log |f_\lambda(e^{i\theta})| \frac{d\theta}{2\pi} = |f_\lambda(0)| \prod_{|z_j| < \lambda} \frac{\lambda}{z_j}.
\]

Now just note that \( |f_\lambda(0)| = |F(0)| S(0) \prod_{j > 0} |z_j| \). This last lemma is straightforward.
Lemma 2.4. If the function $f$ in $\mathcal{A}_p$ has concentration $d$ at degree $k$ and $0 < \lambda \leq 1$, then

$$(i) \; d \| f_\lambda \|_p \leq \| s_k(f_\lambda) \|_p \quad \text{and} \quad (ii) \; \lambda^k d \| f \|_p \leq \| s_k(f_\lambda) \|_p.$$ 

Proof. Consider $f(z) = \sum_{j \geq 0} a_j z^j$ in $\mathcal{A}_p$ with appropriate concentration. Define the function $h : (0, 1] \to \mathbb{R}$ by

$$h(\lambda) = \frac{\| f_\lambda \|_p}{\| s_k(f_\lambda) \|_p} \equiv 1 + \frac{\sum_{j=k+1}^{\infty} |a_j|^p \lambda^{j-pk}}{\sum_{j=0}^{k} |a_j|^p \lambda^{j-pk}}.$$ 

Since $h$ is an increasing function, $h(\lambda) \leq h(1) \leq d^{-p}$. This provides $(i)$. Inequality $(ii)$ follows from the observation that

$$\| s_k(f_\lambda) \|_p = \sum_{j=0}^{k} |a_j|^p \lambda^{j-p} \geq \sum_{j=0}^{k} |a_j|^p \lambda^{kp} = \lambda^{kp} \| s_k(f) \|_p \geq \lambda^{kp} d^p \| f \|_p.$$ 

The proof of Theorem 2.1 now proceeds with ease.

Proof of Theorem 2.1. Since $1 - x \leq -\log(x)$ for $0 < x \leq 1$, the second inequality follows from the first. Towards the first inequality, let the function $f \in \mathcal{A}_p$ have concentration $d$ at degree $k$. We assume that $f$ has more than $k$ zeros for otherwise the theorem is vacuously true. Let the $k^{th}$ zero $z_k$ of $f$ have modulus $\lambda$. The lemmas provide the following string of inequalities

$$\Phi_{d,k} \lambda^k d \| f \|_p \leq \Phi_{d,k} \| s_k(f_\lambda) \|_p \leq \Phi_{d,k} \| f_\lambda \|_p \leq M(f_\lambda) = \lambda^k |F(0)| S(0) \prod_{j > k} |z_j|.$$ 

As noted in the introduction, $S(0) \leq 1$ and $|F(0)| \leq \| f \|_p$. Thus

$$d \Phi_{d,k} \leq \prod_{j > k} |z_j|,$$

as needed.

We do not believe that the bound in Theorem 2.1 is best possible. An improvement in the constant $\Phi_{d,k}$ in Lemma 2.2 would improve the bound in Theorem 2.1.

3. General Comments

In general, it is not possible to bound from below the whole product $\prod_{j > 0} |z_j|$ of zeros $\{z_j\}_{j > 0}$ of a function in $\mathcal{A}_p$ by a constant depending on the function’s concentration at degree $k$ for $k > 0$. For example, consider the function $f(z) = (z^n + 1)(z^n - \epsilon)$ where $0 < \epsilon < 1$ and $n$ is a positive integer. The whole product $\prod_{j > 0} |z_j|$ of zeros is $\epsilon$ yet $f$ has concentration $\frac{1}{8}$ (measured in $\ell_p$-norm) at degree $n$. Information on the zeros of $s_k(f)$ proves useful in this setting.
Theorem 3.1. If the function \( f \) in \( \mathcal{A}_p \) has concentration \( d \) at degree \( k \), then the zeros \( \{z_j\}_{j>0} \) of \( f \) and the zeros \( \{w_i\}_{i>0} \) of \( s_k(f) \) in the unit disk \( D \) satisfy

\[
\frac{d}{2^k} \leq \frac{\prod_{j>0} |z_j|}{\prod_{i>0} |w_i|}.
\]

Thus if in the above setting \( s_k(f) \) has no zeros in the unit disk \( D \), then

\[
\frac{d}{2^k} \leq \prod_{j>0} |z_j|.
\]

Proof. Consider a function \( f \) in \( \mathcal{A}_p \) with the appropriate concentration, i.e.

\[
d \parallel f \parallel_p \leq \parallel s_k(f) \parallel_p.
\]

From Jensen’s Formula it follows [cf. M] that for the polynomial \( s_k(f) : D \to \mathbb{C} \) we have

\[
\frac{\|s_k(f)\|_p}{2^k} \leq M(s_k(f)) = \left| \frac{s_k(f)(0)}{\prod_{i>0} |w_i|} \right|.
\]

Basic properties of the canonical factorization \( f = F \cdot B \cdot S \) of \( f \) give that

\[
\frac{|f(0)|}{\prod_{j>0} |z_j|} = |F(0)| S(0) \leq |F(0)| \leq \|F\|_2 = \|f\|_2 \leq \|f\|_p.
\]

The theorem now follows. \( \blacksquare \)

We may view any function \( f \in \mathcal{H}^p \) as a function in the classical Lebesgue space \( L^p \) (\( T \)) with norm \( \|f\|_p \). Beauzamy and Enflo [BE, Cor 7] showed that if the polynomials \( f_1 \) and \( f_2 \) in \( \mathcal{A}_2 \) have concentration \( d_i \) at degree \( k_i \) (respectively), then

\[
\frac{d_1^2 d_2^2}{3e^{15} \mathbb{S}^{(k_1+k_2+2)}} \parallel f_1 \parallel_2 \parallel f_2 \parallel_2 \leq \|f_1 f_2\|_1.
\]

The following theorem improves on this result.

Theorem 3.2. Fix \( 1 \leq p_1, p_2 \leq 2 \). If \( f_i \in \mathcal{A}_{p_i} \) has concentration \( d_i \) at degree \( k_i \), then

\[
\Phi_{d_1, k_1} \Phi_{d_2, k_2} \parallel f_1 \parallel_{p_1} \parallel f_2 \parallel_{p_2} \leq M(f_1 f_2).
\]

Theorem 3.2 follows directly from Lemma 2.2 and the observation that \( M(f_1) M(f_2) = M(f_1 f_2) \). Recall that for any function \( g \in \mathcal{H}^2 \)

\[
M(g) \leq \|g\|_1.
\]

Also

\[
\frac{d_1^2 d_2^2}{3e^{15} \mathbb{S}^{(k_1+k_2+2)}} < \frac{16 d_1^2 d_2^2}{e^4 \mathbb{S}^{(k_1+k_2+2)}} = \phi_{d_1, k_1} \left( \frac{1}{3} \right) \phi_{d_2, k_2} \left( \frac{1}{3} \right) \leq \Phi_{d_1, k_1} \Phi_{d_2, k_2}.
\]
Thus Theorem 3.2 gives (with a better constant) the result of Beauzamy and Enflo.

It is well-known that there exists positive constants $k_n^p$ and $K_n^p$ such that if $f$ is a polynomial of degree $n$ then

$$k_n^p \| f \|_p \leq M(f) \leq K_n^p \| f \|_p.$$  

Lemma 2.2 gives that if $f \in A_p$ has concentration $d$ at degree $k$, then

$$\Phi_{d,k} \| f \|_p \leq M(f) \leq \| f \|_p.$$  

Thus Lemma 2.2 may be thought of as extending these well-known inequality from polynomials (with a certain degree) to $A_p$ functions (with a certain concentration).

Of course, the settings of most interest are $A_1$ and $A_2$. We worked in $A_p$ with $1 \leq p \leq 2$ as to provide unity in presentation. For $p > 2$, the definitions are valid but then the space $A_p$ is not as natural; the $\sum |a_j|^p$ being finite implies that the corresponding function $f(z) = \sum_{j \geq 0} a_j z^j$ is in $H^p$ if and only if $1 \leq p \leq 2$.

Our method of proof does not extend directly to the case of $p > 2$. Our proofs use the fact that if $1 \leq p \leq 2$ and $f$ is in $A_p$ then $M(f) \leq \| f \|_p$. However, this basic inequality is not valid for $p > 2$, even in a wider sense. Specifically [cf. BN], there is a sequence of polynomials $\{f_n\}$ such that $M(f_n) \sim \sqrt{n}$ and $f_n$ is of degree $n$ with coefficients of modulus 1 (thus $\| f_n \|_p = n + 1$). For this sequence of polynomials, $\| f_n \|_{M(f_n)}$ tends to 0.

This sequence also demonstrates that, for $p > 2$, one does not even have the motivating bound (dependent on the concentration at degree $k = 0$) on the product of the zeros of a function in $A_p$. For since $|f_n(0)| = 1$, the modulus of the product of the zeros of $f_n$ is $[M(f_n)]^{-1}$, which grows like $n^{-1/2}$. However, at degree $k = 0$, the function $f_n$ has concentration $[n + 1]^{-1/p}$.

References


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