BOUNDING ZEROS OF H^2 FUNCTIONS VIA CONCENTRATIONS

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J. Math. Anal. Appl. **183** *(1994) 605–612*

Abstract. It is well-known that the zeros $\{z_j\}$ of a function in the classical Hardy space H^2 satisfy $\sum 1-|z_j| < \infty$; however, this sum can be arbitrarily large. We shall bound this sum by a constant that depends on the concentration of the function, a concept introduced by Beauzamy and Enflo.

1. INTRODUCTION

Consider a function $f : D \to \mathbf{C}$ in the classical Hardy space $H^2(D)$ where D is the open unit disk in the complex plane $\mathbf{C}$. It is well-known that the zeros $\{z_j\}$ of f satisfy $\sum 1-|z_j| < \infty$. However, this sum can be arbitrarily large as seen by considering an appropriate Blaschke product.

Fix $1 \le p \le 2$ and consider the subset $\mathcal{A}_p$ of H^2 where

$$\mathcal{A}_p = \{f \in H^p(D) : f(z) = \sum_{j\ge 0} a_j z^j \quad \text{and} \quad \sum_{j\ge 0} |a_j|^p < \infty\}.$$

Of course, $\mathcal{A}_2$ is just H^2 and $\mathcal{A}_1$ is the usual algebra $\mathcal{A}_+(T)$. For a function in $\mathcal{A}_p$ with zeros $\{z_j\}$, we shall bound $\sum 1-|z_j|$ by a constant that depends on the concentration of the function, a notation introduced by Beauzamy and Enflo [BE].

Now to recall basic facts and fix notation. For

$$f(z) = \sum_{j\ge 0} a_j z^j \in \mathcal{A}_p,$$

the l_p-norm of f is

$$\| f \|_p = \left[\sum_{j\ge 0} |a_j|^p \right]^{1/p},$$

Supported in part by the C.N.R.S. (France) and the N.S.F. (U.S.A.), by contracts D.G.A./-D.R.E.T. no 89/1377, E.T.C.A./C.R.E.A./20367/91 (Ministry of Defense, France) and by research contract EERP-FR 22, *Digital Equipment Corporation*. Research conducted while at the Institut de Calcul Mathématique, Université de Paris VII, 2 place Jussieu, 75251 Paris Cedex 05, France.

and the k-th partial sum $s_k(f)$ of f is

$$[s_k(f)](z) = \sum_{j=0}^{k} a_j z^j .$$

The function $f \in \mathcal{A}_p$ has concentration d at degree k (measured in ℓ_p-norm) if

$$d \;\; \| f \|_p \;\; \leq \| s_k(f) \|_p$$

where $0 < d \leq 1$ and k is a non-negative integer. The largest d which satisfies the above condition is called the concentration factor.

An H^2 function f has a canonical factorization $f(z) \;=\; F(z)B(z)S(z)$ into an outer function F, a Blaschke product B, and a singular part S. The measure $M(f)$ of f is

$$M(f) \;=\; \exp\left[\int_0^{2\pi} \log|f(e^{i\theta})| \; \frac{d\theta}{2\pi}\right].$$

Recall $M(f) \;=\; |F(0)| \;\; \leq \| \; F \; \|_2 \; = \; \| \; f \; \|_2$ and since $1 \leq p \leq 2$ furthermore $\| \; f \; \|_2 \leq \| \; f \; \|_p$.

For $0 \leq \lambda \leq 1$, let $f_\lambda : D \to \mathbf{C}$ be the function $f_\lambda(z) \;=\; f(\lambda z)$. Throughout this paper, we will assume that $f(0) \neq 0$ and enumerate the zero set $\{z_j\}$ of f so that

$$0 < |z_1| \leq |z_2| \leq \ldots < 1.$$

All notation and terminology, not otherwise explained, are as in [G] or [B2].

As motivation we examine a function $f \in \mathcal{A}_p$ with concentration d at degree $k = 0$. For such a function we have that

$$d \;\; \| f \|_p \leq \| s_0(f) \|_p \; = \; |f(0)| \;\; = \;\; |F(0)| \; S(0) \prod_{j>0} |z_j|.$$

We know that the singular part S of an H^2 function satisfies $0 \leq S(0) \leq 1$. Thus

$$d \;\leq\; \prod_{j>0} |z_j|,$$

and since $1 - x \;\; \leq \;\; -\log x$ for $0 < x \leq 1$, we see that

$$\sum_{j>0} 1 - |z_j| \leq -\log d \; .$$

The first bound is essentially best possible. Consider the family of H^2 functions $\{f_\varepsilon(z) = z - \varepsilon \;\; : 0 < \varepsilon < 1\}$. For $1 \leq p \leq 2$, the function f_ε has concentration $d \;=\; \frac{\varepsilon}{1+\varepsilon}$ at degree 0. Thus

$$d \;\leq\; \prod_{j>0} |z_j| \;\;=\;\; \varepsilon \;\;\leq\;\; d(1+\varepsilon).$$

We now extend these ideas for concentration at an arbitrary degree k.

2. ZEROS OF FUNCTIONS IN $\mathcal{A}_p$

Beauzamy and Chou [BC] showed that the zeros $\{z_j\}$ of a polynomial with concentration d at degree k (measured in ℓ_1-norm) satisfy

$$\frac{d^{k+3}}{2\ 9^{k+3}\ 3^{k^2}} \ \leq\ \prod_{j>k} |z_j|\ .$$

Actually, their product includes the zeros outside the unit disk as well. Theorem 2.1 extends this result to functions in $\mathcal{A}_p$ and improves their constant. For our constant, we consider the function

$$\phi_{d,k}(r)\ \equiv\ \left[\frac{(1-r)r^k d}{e^{\frac{1+r}{2(1-r)}}}\right]^{\frac{1+r}{1-r}}$$

and let

$$\Phi_{d,k}\ \equiv\ \max_{0<r<1} \phi_{d,k}(r)\ .$$

Recall that we enumerate a zero set $\{z_j\}$ so that $0<|z_1|\leq|z_2|\leq\ldots<1$. We now fix $1\leq p\leq 2$.

Theorem 2.1. *The zeros $\{z_j\}$ of a function in $\mathcal{A}_p$ with concentration d at degree k (measured in ℓ_p - norm) satisfy*

$$d\ \Phi_{d,k}\ \leq\ \prod_{j>k}|z_j| \qquad \textit{and thus} \qquad \sum_{j>0} 1-|z_j|\ \leq\ k-\log\left(d\ \Phi_{d,k}\right)\ .$$

To compare with [BC], note that $\frac{d^{k+3}}{2\ 9^{k+3}\ 3^{k^2}}\ <\ \frac{4d^3}{e^2 9^{k+1}}\ =\ d\ \phi_{d,k}(\frac{1}{3})\ \leq\ d\ \Phi_{d,k}$.

The proof uses the following lemma which extends a result of Beauzamy ([B1], [B2]).

Lemma 2.2. *If a function f in $\mathcal{A}_p$ has concentration d at degree k, then*

$$\Phi_{d,k}\,\|\ f\ \|_p\ \leq\ M(f)\ .$$

As noted earlier $M(f)\ \leq \|\ f\ \|_p$ since $1\leq p\leq 2$.

Proof. Let $f(z)=\sum_{j\geq 0} a_j z^j$ be a function in $\mathcal{A}_p$ with appropriate concentration. Normalize so that $\|\ f\ \|_p =\ 1$. Fix $0<r<1$ and consider $z_0\in D$ with $|z_0|=r$ along with its Möbius function $w(z)\ =\ \frac{z+z_0}{1+\bar{z}_o z}$ defined on $\overline{D}$. Jensen's formula and a change of variables give that

$$\begin{aligned}
\log|f(z_0)|\ &\leq\ \int_0^{2\pi}\log|f(w(e^{i\theta}))|\ \frac{d\theta}{2\pi}\\
&=\ \int_0^{2\pi}\log|f(e^{i\theta})|\ \frac{1-r^2}{|1-\bar{z}_0 e^{i\theta}|^2}\ \frac{d\theta}{2\pi}\\
&\leq \frac{1-r}{1+r}\ \int_N \log|f(e^{i\theta})|\ \frac{d\theta}{2\pi}\ +\ \frac{1+r}{1-r}\ \int_P \log|f(e^{i\theta})|\ \frac{d\theta}{2\pi}\ ,
\end{aligned}$$

where

$$N = \{ \theta : |f(e^{i\theta})| \le 1\} \qquad \text{and} \qquad P = \{ \theta : |f(e^{i\theta})| > 1\} .$$

For bounding the last summand, note that

$$\int_P \log|f(e^{i\theta})| \frac{d\theta}{2\pi} \le \frac{1}{2}\int |f|^2 \frac{d\theta}{2\pi} = \frac{1}{2} \parallel f \parallel_2^2 \le \frac{1}{2} \parallel f \parallel_p^2 = \frac{1}{2} .$$

Choose z_0 such that $|f(z_0)| = \max \{|f(z)| : |z| = r\}$. Since

$$a_j = \int_0^{2\pi} \frac{f(re^{i\theta})}{r^j e^{ij\theta}} \frac{d\theta}{2\pi} \qquad \text{and so} \qquad |a_j| \le \frac{|f(z_0)|}{r^j}$$

it follows that

$$\begin{aligned} \parallel s_k(f) \parallel_p \le \parallel s_k(f) \parallel_1 &= \sum_{j=0}^{k} |a_j| \le |f(z_0)| \sum_{j=0}^{k} \frac{1}{r^j} \\ &\le |f(z_0)|\, r^{-k} \sum_{j=0}^{\infty} r^j = \frac{|f(z_0)|}{(1-r)r^k} . \end{aligned}$$

Combining this with the concentration information on f we see that

$$(1-r)r^k d \le | f(z_0) | .$$

We now have that

$$\log\left((1-r)r^k d\right) - \frac{1+r}{2(1-r)} \le \frac{1-r}{1+r} \int_N \log | f(e^{i\theta}) | \frac{d\theta}{2\pi} ,$$

which gives that $\phi_{d,k}(r) \le M(f)$, as needed. ■

Keeping with the previous notation, we observe a property of the measure.

Lemma 2.3. *If the H^2 function f has a k-th zero of modulus λ, then*

$$M(f_\lambda) = \lambda^k |F(0)|\, S(0) \prod_{j>k} |z_j| .$$

Proof. Since f_λ is analytic on the closed unit disk $\overline{D}$, Jensen's formula provides that

$$M(f_\lambda) = \exp \int_0^{2\pi} \log|f_\lambda(e^{i\theta})| \frac{d\theta}{2\pi} = | f_\lambda(0) | \prod_{|z_j|<\lambda} \frac{\lambda}{| z_j |} .$$

Now just note that $|f_\lambda(0)| = |F(0)|\, S(0) \prod_{j>0} |z_j|$. ■

This last lemma is straightforward.

Lemma 2.4. *If the function f in $\mathcal{A}_p$ has concentration d at degree k and $0 < \lambda \leq 1$, then*

$$(i)\ d \parallel f_\lambda \parallel_p \leq \parallel s_k(f_\lambda) \parallel_p \qquad \textit{and} \qquad (ii)\ \lambda^k d \parallel f \parallel_p \leq \parallel s_k(f_\lambda) \parallel_p \ .$$

Proof. Consider $f(z) = \sum_{j\geq 0} a_j z^j$ in $\mathcal{A}_p$ with appropriate concentration. Define the function $h : (0,1] \to \mathbb{R}$ by

$$h(\lambda) = \frac{\parallel f_\lambda \parallel_p^p}{\parallel s_k(f_\lambda) \parallel_p^p} \quad \equiv \quad 1 \quad + \quad \frac{\sum_{j=k+1}^{\infty} |a_j|^p \lambda^{pj-pk}}{\sum_{j=0}^{k} |a_j|^p \lambda^{pj-pk}} \ .$$

Since h is an increasing function, $h(\lambda) \leq h(1) \leq d^{-p}$. This provides (i). Inequality (ii) follows from the observation that

$$\parallel s_k(f_\lambda) \parallel_p^p = \sum_{j=0}^{k} |a_j|^p \ \lambda^{jp} \ \geq \ \sum_{j=0}^{k} |a_j|^p \ \lambda^{kp} \ = \ \lambda^{kp} \ \parallel s_k(f) \parallel_p^p \geq \ \lambda^{kp} d^p \parallel f \parallel_p^p \ .$$

■

The proof of Theorem 2.1 now proceeds with ease.

Proof of Theorem 2.1. Since $1 - x \leq -\log(x)$ for $0 < x \leq 1$, the second inequality follows from the first. Towards the first inequality, let the function $f \in \mathcal{A}_p$ have concentration d at degree k. We assume that f has more than k zeros for otherwise the theorem is vacuously true. Let the k^{th} zero z_k of f have modulus λ. The lemmas provide the following string of inequalities

$$\begin{aligned} \Phi_{d,k}\ \lambda^k\ d \parallel f \parallel_p \ &\leq \ \Phi_{d,k} \ \parallel s_k(f_\lambda) \parallel_p \\ &\leq \ \Phi_{d,k} \ \parallel f_\lambda \parallel_p \leq \ M(f_\lambda) \ = \ \lambda^k |F(0)| \ S(0) \prod_{j>k} |z_j| \ . \end{aligned}$$

As noted in the introduction, $S(0) \leq 1$ and $|F(0)| \ \leq \parallel f \parallel_p$. Thus

$$d \ \Phi_{d,k} \ \leq \ \prod_{j>k} |z_j| \ ,$$

as needed. ■

We do not believe that the bound in Theorem 2.1 is best possible. An improvement in the constant $\Phi_{d,k}$ in Lemma 2.2 would improve the bound in Theorem 2.1.

3. General Comments

In general, it is not possible to bound from below the whole product $\prod_{j>0} |z_j|$ of zeros $\{z_j\}_{j>0}$ of a function in $\mathcal{A}_p$ by a constant depending on the function's concentration at degree k for $k > 0$. For example, consider the function $f(z) = (z^n + 1)(z^n - \epsilon)$ where $0 < \epsilon < 1$ and n is a positive integer. The whole product $\prod_{j>0} |z_j|$ of zeros is ϵ yet f has concentration $\frac{1}{8}$ (measured in ℓ_p-norm) at degree n. Information on the zeros of $s_k(f)$ proves useful in this setting.

Theorem 3.1. *If the function f in $\mathcal{A}_p$ has concentration d at degree k, then the zeros $\{z_j\}_{j>0}$ of f and the zeros $\{w_i\}_{i>0}$ of $s_k(f)$ in the unit disk D satisfy*

$$\frac{d}{2^k} \leq \frac{\prod_{j>0} |z_j|}{\prod_{i>0} |w_i|} .$$

Thus if in the above setting $s_k(f)$ has no zeros in the unit disk D, then

$$\frac{d}{2^k} \leq \prod_{j>0} |z_j|.$$

Proof. Consider a function f in $\mathcal{A}_p$ with the appropriate concentration, i.e.

$$d \ \| f \|_p \leq \| s_k(f) \|_p \ .$$

From Jensen's Formula it follows [cf. M] that for the polynomial $s_k(f) \ : D \to \mathbf{C}$ we have

$$\frac{\| s_k(f) \|_p}{2^k} \leq M(s_k(f)) = \frac{|s_k(f)(0)|}{\prod_{i>0} |w_i|} .$$

Basic properties of the canonical factorization $f = F \cdot B \cdot S$ of f give that

$$\frac{|f(0)|}{\prod_{j>0} |z_j|} = |F(0)| \ S(0) \ \leq | \ F(0) \ | \leq \| \ F \ \|_2 = \| \ f \ \|_2 \leq \| \ f \ \|_p \ .$$

The theorem now follows. ■

We may view any function $f \in H^p$ as a function in the classical Lebesgue space $L^p \ (T)$ with norm $|||f|||_p$. Beauzamy and Enflo [BE, Cor 7] showed that if the polynomials f_1 and f_2 in $\mathcal{A}_2$ have concentration d_i at degree k_i (respectively), then

$$\frac{d_1^6 \, d_2^6}{3e^{15} 9^{3(k_1+k_2+2)}} \ \| \ f_1 \ \|_2 \ \| \ f_2 \ \|_2 \leq \ |||f_1 \ f_2|||_1 \ .$$

The following theorem improves on this result.

Theorem 3.2. *Fix $1 \leq p_1, p_2 \leq 2$. If $f_i \in \mathcal{A}_{p_i}$ has concentration d_i at degree k_i, then*

$$\Phi_{d_1,k_1} \ \Phi_{d_2,k_2} \ \| \ f_1 \ \|_{p_1} \ \| \ f_2 \ \|_{p_2} \leq \ M(f_1 f_2) \ .$$

Theorem 3.2 follows directly from Lemma 2.2 and the observation that $M(f_1)M(f_2) = M(f_1 f_2)$. Recall that for any function $g \in H^2$

$$M(g) \leq |||g|||_1 \ .$$

Also

$$\frac{d_1^6 \, d_2^6}{3e^{15} 9^{3(k_1+k_2+2)}} < \frac{16 \ d_1^2 d_2^2}{e^4 \ 9^{k_1+k_2+2}} = \phi_{d_1,k_1}\left(\tfrac{1}{3}\right) \phi_{d_2,k_2}\left(\tfrac{1}{3}\right) \leq \Phi_{d_1,k_1} \ \Phi_{d_2,k_2} \ .$$

Thus Theorem 3.2 gives (with a better constant) the result of Beauzamy and Enflo.

It is well-known that there exists positive constants k_n^p and K_n^p such that if f is a polynomial of degree n then

$$k_n^p \ \| f \|_p \leq \ M(f) \ \leq \ K_n^p \| f \|_p \ .$$

Lemma 2.2 gives that if $f \in \mathcal{A}_p$ has concentration d at degree k, then

$$\Phi_{d,k} \ \| f \|_p \leq \ M(f) \ \leq \| f \|_p \ .$$

Thus Lemma 2.2 may be thought of as extending these well-known inequality from polynomials (with a certain degree) to $\mathcal{A}_p$ functions (with a certain concentration).

Of course, the settings of most interest are $\mathcal{A}_1$ and $\mathcal{A}_2$. We worked in $\mathcal{A}_p$ with $1 \leq p \leq 2$ as to provide unity in presentation. For $p > 2$, the definitions are valid but then the space $\mathcal{A}_p$ is not as natural; the $\sum |a_j|^p$ being finite implies that the corresponding function $f(z) \ = \ \sum_{j \geq 0} a_j z^j$ is in H^p if and only if $1 \leq p \leq 2$.

Our method of proof does not extend directly to the case of $p > 2$. Our proofs use the fact that if $1 \leq p \leq 2$ and f is in $\mathcal{A}_p$ then $M(f) \ \leq \| f \|_p$. However, this basic inequality is not valid for $p > 2$, even in a wider sense. Specifically [cf. BN], there is a sequence of polynomials $\{f_n\}$ such that $M(f_n) \sim \sqrt{n}$ and f_n is of degree n with coefficients of modulus 1 (thus $\| \ f_n \ \|_p^p = \ n+1$). For this sequence of polynomials, $\frac{\|f_n\|_p}{M(f_n)}$ tends to 0.

This sequence also demonstrates that, for $p > 2$, one does not even have the motivating bound (dependent on the concentration at degree $k = 0$) on the product of the zeros of a function in $\mathcal{A}_p$. For since $|f_n(0)| = 1$, the modulus of the product of the zeros of f_n is $[M(f_n)]^{-1}$, which grows like $n^{-1/2}$. However, at degree $k = 0$, the function f_n has concentration $[n+1]^{-1/p}$.

References

[B1]. Bernard Beauzamy, *Jensen's inequality for polynomials with concentration at low degrees*, Numer. Math. **49** (1986), 221–225.

[B2]. Bernard Beauzamy, *Estimates for H^2 functions with concentration at low degrees and applications to complex symbolic computations*, J. Reine Angew. Math. **433** (1992).

[BC]. Bernard Beauzamy and Sylvia Chou, *On the zeros of polynomials with concentration at low degrees*, (to appear).

[BE]. Bernard Beauzamy and Per Enflo, *Estimations de produits de polynômes*, J. Number Theory **21** (1985), 390–412.

[BN]. E. Beller and D. J. Newman, *An extremal problem for the geometric mean of polynomials*, Proc. Amer. Math. Soc. **39** (1973), 313–317.

[G]. John B. Garnett, *Bounded Analytic Functions*, Academic Press, New York, 1981.

[M]. K. Mahler, *An application of Jensen's formula to polynomials*, Mathematika (1960), 98–100.

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