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Bochner vs. Pettis norm: examples and results

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Abstract. Our basic example shows that for an *arbitrary* infinite-dimensional Banach space $\mathfrak{X}$, the Bochner norm and the Pettis norm on $L_1(\mathfrak{X})$ are not equivalent. Refinements of this example are then used to investigate various modes of sequential convergence in $L_1(\mathfrak{X})$.

1. INTRODUCTION

Over the years, the Pettis integral along with the Pettis norm have grabbed the interest of many. In this note, we wish to clarify the differences between the Bochner and the Pettis norms. We begin our investigation by using Dvoretzky's Theorem to construct, for an *arbitrary* infinite-dimensional Banach space, a sequence of Bochner integrable functions whose Bochner norms tend to infinity but whose Pettis norms tend to zero. By refining this example (again working with an arbitrary infinite-dimensional Banach space), we produce a Pettis integrable function that is not Bochner integrable and we show that the space of Pettis integrable functions is not complete. Thus our basic example provides a unified constructive way of seeing several known facts. The third section expresses these results from a vector measure viewpoint. In the last section, with the aid of these examples, we give a fairly thorough survey of the implications going between various modes of convergence for sequences of $L_1(\mathfrak{X})$ functions.

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2. EXAMPLES: $L_1(\mathfrak{X})$ vs. $P_1(\mathfrak{X})$

Let $\mathfrak{X}$ be a Banach space with dual $\mathfrak{X}^*$ and let (Ω, Σ, μ) be the usual Lebesgue measure space on $[0,1]$. Let $L_1(\mathfrak{X}) = L_1(\Omega, \Sigma, \mu; \mathfrak{X})$ be the space of $\mathfrak{X}$-valued Bochner integrable functions on Ω with the usual Bochner norm $||\cdot||_{\text{Bochner}}$.

A quick review of the Pettis integral. Consider a weakly measurable function $f : \Omega \to \mathfrak{X}$ such that $x^*f \in L_1(\mathbb{R})$ for all $x^* \in \mathfrak{X}^*$. The Closed Graph Theorem gives that for each $E \in \Sigma$ there is an element x_E^{**} of $\mathfrak{X}^{**}$ satisfying $x_E^{**}(x^*) = \int_E x^* f d\mu$ for each $x^* \in \mathfrak{X}^*$. If x_E^{**} is actually in $\mathfrak{X}$ for each $E \in \Sigma$, then we say that f is Pettis integrable with the Pettis integral of f over E being the element $x_E \in \mathfrak{X}$ satisfying

$$x^*(x_E) = \int_E x^* f d\mu$$

for each $x^* \in \mathfrak{X}^*$. In this note we shall restrict our attention to Pettis integrable functions that are strongly measurable. The space $\mathcal{P}_1(\mathfrak{X})$ of (equivalence classes of) all strongly measurable Pettis integrable functions forms a normed linear space under the norm

$$\| f \|_{\text{Pettis}} = \sup_{x^* \in B(\mathfrak{X}^*)} \int_\Omega | x^*(f) | \, d\mu \ .$$

Clearly if $f \in L_1(\mathfrak{X})$ then $f \in \mathcal{P}_1(\mathfrak{X})$ with

$$||f||_{\text{Pettis}} \leq ||f||_{\text{Bochner}} \ .$$

If $\mathfrak{X}$ is finite-dimensional, then the Bochner norm and the Pettis norm are equivalent and so $L_1(\mathfrak{X}) = \mathcal{P}_1(\mathfrak{X})$. However, in *any* infinite-dimensional Banach space $\mathfrak{X}$, the next example shows that the two norms are not equivalent on $L_1(\mathfrak{X})$. Specifically, it constructs an *essentially bounded* sequence of Bochner-norm one functions whose Pettis norms tend to 0.

EXAMPLE 1.

Let $\mathfrak{X}$ be an infinite-dimensional Banach space. We now construct a sequence $\{f_n\}$ of $L_1(\mathfrak{X})$ functions which tend to zero in the Pettis norm but not in the Bochner norm.

Let $\{I_k^n : n = 0, 1, \dots ; k = 1, \dots, 2^n\}$ be the dyadic intervals on $[0,1]$, i.e.

$$I_k^n = \left[\tfrac{k-1}{2^n}, \tfrac{k}{2^n}\right) \ .$$

Fix $n \in \mathbb{N}$. We define f_n with the help of Dvoretzky's Theorem [D]. Find a 2^n-dimensional subspace E_n of $\mathfrak{X}$ such that the Banach-Mazur distance between E_n and $\ell_2^{2^n}$ is at most 2. So there is an operator $T_n \colon \ell_2^{2^n} \to E_n$ such that the norm of T_n is at most 2 and the norm of the inverse of T_n is 1. Let $\{e_k^n \colon k = 1, \dots 2^n\}$

be the image under T_n of the standard unit vectors $\{u_k^n : k = 1, \ldots 2^n\}$ of $\ell_2^{2^n}$. Define $f_n : \Omega \to \mathfrak{X}$ by

$$f_n(\omega) \;=\; \sum_{k=1}^{2^n} e_k^n \; 1_{I_k^n}(\omega) \;,$$

The sequence $\{f_n\}$ has the desired properties.

Since

$$||f_n||_{\text{Bochner}} \;=\; \sum_{k=1}^{2^n} \int_{I_k^n} ||e_k^n||_{\mathfrak{X}} \, d\mu \;=\; 2^{-n} \sum_{k=1}^{2^n} ||e_k^n||_{\mathfrak{X}}$$

and $1 \leq ||e_k^n||_{\mathfrak{X}} \leq 2$,

$$1 \;\leq\; ||f_n||_{\text{Bochner}} \;\leq\; 2 \;.$$

As for the Pettis norm, fix $x^* \in B(X^*)$. Note that the restriction y_n^* of x^* to E_n is an element in E_n^* of norm at most 1. Also, the adjoint T_n^* of T_n has norm at most 2. Thus

$$\begin{aligned} \sum_{k=1}^{2^n} |x^*(e_k^n)| \;&=\; \sum_{k=1}^{2^n} |y_n^*(T_n u_k^n)| \\ &=\; \sum_{k=1}^{2^n} |(T_n^* y_n^*)(u_k^n)| \;\leq\; ||T_n^* y_n^*||_{(\ell_2^{2^n})^*} \; ||\sum_{k=1}^{2^n} u_k^n||_{\ell_2^{2^n}} \;\leq\; 2\,\sqrt{2^n} \;, \end{aligned}$$

and so

$$\int_\Omega |x^*(f_n)| \, d\mu \;=\; \sum_{k=1}^{2^n} \int_{I_k^n} |x^*(e_k^n)| \, d\mu \;=\; 2^{-n} \sum_{k=1}^{2^n} |x^*(e_k^n)| \;\leq\; 2 \, 2^{-\frac{n}{2}} \;,$$

giving that

$$||f_n||_{\text{Pettis}} \;\leq\; 2 \, 2^{-\frac{n}{2}} \;.$$

Thus the Pettis norm of f_n tends to zero. $\square$

This difference between the Bochner and Pettis norms is further emphasized by the following minor variation of Example 1.

EXAMPLE 2.

Fix $0 < \beta < \frac{1}{2}$ and put $\alpha_n \equiv 2^{n\beta}$. Define $\tilde{f}_n : \Omega \to \mathfrak{X}$ by $\tilde{f}_n \;=\; \alpha_n f_n$ where f_n is as in Example 1. The Bochner norm of $\tilde{f}_n$ tends to infinity at the rate of α_n but the Pettis norm of $\tilde{f}_n$ tends to zero. $\square$

A variation on this example gives a (strongly measurable) Pettis integrable function that is not Bochner integrable – working in *any* infinite-dimensional Banach space. The usual way of constructing such a function uses the Dvoretsky-Rogers Theorem. Recall [DR] that in any infinite-dimensional Banach space $\mathfrak{X}$, there is an unconditionally convergent series $\sum_n x_n$ that is not absolutely convergent. The function $f : \Omega \to \mathfrak{X}$ given by

$$f(\cdot) \;=\; \sum_{n=1}^{\infty} \frac{x_n}{\mu(E_n)} \; 1_{E_n}(\cdot),$$

(where $\{E_n\}$ is a partition of Ω into sets of strictly positive measure) is Pettis integrable but not Bochner integrable. Our example is in the same spirit.

EXAMPLE 3.

Let $\mathfrak{X}$ be an infinite-dimensional Banach space. Consider the above construction. Note that the collection $\{I_2^n\}_{n=1}^{\infty}$ partitions $[0,1]$. Let $g_n \colon \Omega \to \mathfrak{X}$ be a normalized f_n (from Example 1) supported on I_2^n, i.e.

$$g_n(\omega) \;=\; 2^n \; f_n(2^n\omega - 1) \; 1_{I_2^n}(\omega) \; .$$

Define $f \colon \Omega \to \mathfrak{X}$ by

$$f(\omega) \;=\; \sum_{n=1}^{\infty} g_n(\omega) \; 1_{I_2^n}(\omega) \; .$$

Clearly f is strongly measurable.

But f is not Bochner integrable since for any $N \in \mathbb{N}$

$$\int_{\Omega} ||f|| \, d\mu \quad \geq \quad \sum_{n=1}^{N} \int_{I_2^n} ||g_n||_{\mathfrak{X}} \, d\mu \quad \geq \quad \sum_{n=1}^{N} 2^{-n} \, 2^n \quad = \quad N \; .$$

However f is Pettis integrable. First note that $x^* f \in L_1(\mathbb{R})$ for all $x^* \in \mathfrak{X}^*$ since for a fixed $x^* \in B(X^*)$ we have (using computations from the first example)

$$\int_{I_2^n} |x^* f| \, d\mu \;\;=\;\; \int_{I_2^n} |x^* g_n| \, d\mu \;\;=\;\; \int_{\Omega} |x^* f_n| \, d\mu \;\leq\; 2 \; 2^{-\frac{n}{2}} \; ,$$

and so

$$\int_{\Omega} |x^* f| \, d\mu \;\;=\;\; \sum_{n=1}^{\infty} \int_{I_2^n} |x^* f| \, d\mu \;\;\leq\;\; 2 \sum_{n=1}^{\infty} 2^{-\frac{n}{2}} \;\;=\;\; 2(\sqrt{2}+1) \; .$$

Next fix $E \in \Sigma$. We know there is an element x_E^{**} of $\mathfrak{X}^{**}$ satisfying that $x_E^{**}(x^*) \;=\; \int_E x^* f \, d\mu$ for each $x^* \in \mathfrak{X}^*$. To see that x_E^{**} is actually in $\mathfrak{X}$, consider the sequence $\{s_N\}$ of $L_1(\mathfrak{X})$ functions on Ω where

$$s_N(\omega) \;=\; \sum_{n=1}^{N} g_n(\omega) \; 1_{I_2^n}(\omega) \; .$$

Let x_E^N be the Bochner integral of s_N over E. Viewed as a sequence in $\mathfrak{X}^{**}$, $\{x_E^N\}_N$ converges in norm to x_E^{**} since for any $x^* \in B(X^*)$

$$\begin{aligned} |(x_E^N - x_E^{**})(x^*)| \;&=\; \Big| \int_E x^* s_N \, d\mu - \int_E x^* f \, d\mu \Big| \;\leq\; \int_{\Omega} |x^*(s_N - f)| \, d\mu \\ &=\; \sum_{n=N+1}^{\infty} \int_{I_2^n} |x^* f| \, d\mu \;\;\leq\;\; 2 \sum_{n=N+1}^{\infty} 2^{-\frac{n}{2}} \; . \end{aligned}$$

Thus x_E^{**} is actually in $\mathfrak{X}$, as needed. $\square$

Janicka and Kalton [JK] showed that if $\mathfrak{X}$ is infinite-dimensional then $\mathcal{P}_1(\mathfrak{X})$ is *never* complete under the Pettis norm. The first example provides a constructive way to see this.

EXAMPLE 4.

Taking some care in our construction in Example 1, we may arrange for $\{E_n\}$ to form a finite dimensional decomposition (FDD) of some subspace of $\mathfrak{X}$. To see this, first choose a basic sequence $\{x_n\}$ in $\mathfrak{X}$. Take a blocking $\{F_n\}$ of the basis so that each subspace F_n is of large enough dimension to find (using the finite-dimensional version of Dvoretzky's Theorem) a 2^n-dimensional subspace E_n of F_n such that the Banach-Mazur distance between E_n and $\ell_2^{2^n}$ is at most 2. Then $\{E_n\}$ forms a FDD.

Keeping with the notation from Example 1, define $h_n \colon \Omega \to \mathfrak{X}$ by

$$h_n \; = \; \sum_{k=1}^{n} f_k \; .$$

Clearly each h_n is in $\mathcal{P}_1(\mathfrak{X})$. Also, $\{h_n\}$ is Cauchy in the Pettis norm since

$$||h_n - h_{n+j}||_{\text{Pettis}} \;\; \leq \;\; \sum_{k=n}^{\infty} ||f_k||_{\text{Pettis}} \;\; \leq \;\; 2 \sum_{k=n}^{\infty} 2^{-\frac{k}{2}} \; .$$

To show that h_n cannot converge to an element in $\mathcal{P}_1(\mathfrak{X})$, we need the following two lemmas whose proofs are given shortly.

LEMMA 1. If the sequence $\{f_n\}$ of functions in $\mathcal{P}_1(\mathfrak{X})$ converges to $f \in \mathcal{P}_1(\mathfrak{X})$ and each f_n is essentially valued in some subspace $\mathfrak{Y}$ of $\mathfrak{X}$, then f is also essentially valued in $\mathfrak{Y}$. $\square$

LEMMA 2. Let $\{f_n\}$ be a sequence of $\mathcal{P}_1(\mathfrak{X})$ functions that converges in Pettis norm to $f \in \mathcal{P}_1(\mathfrak{X})$. Furthermore, suppose that each f_n is essentially valued in some subspace $\mathfrak{Y}$ of $\mathfrak{X}$ and that $T : \mathfrak{Y} \to \mathfrak{X}$ is a bounded linear operator. Then Tf_n converges in the Pettis norm to Tf. $\square$

Suppose that h_n converges in the Pettis norm to $h \in \mathcal{P}_1(\mathfrak{X})$. Let $\mathfrak{Y} \equiv \sum \oplus E_n$ and let $P_n \colon \mathfrak{Y} \to \mathfrak{X}$ be the natural projection from $\mathfrak{Y}$ onto E_n. Note that each h_n, thus also h, are essentially valued in $\mathfrak{Y}$. Applying the second lemma to the operator P_n and noting that $P_n h_m \equiv f_n$ for $m \geq n$, we have that $P_n h \equiv f_n$. Thus for each $n \in \mathbb{N}$

$$1 \;\leq\; ||P_n h(\omega)||_{\mathfrak{X}} \;\leq 2$$

for almost all ω. This contradicts the fact that for almost all ω

$$\lim_{N\to\infty} ||h(\omega) - \sum_{n=1}^{N} P_n h(\omega)||_{\mathfrak{X}} \;\; = \;\; 0 \; .$$

Thus h_n cannot converge to an element in $\mathcal{P}_1(\mathfrak{X})$. $\square$

Now for the proofs of the lemmas.

PROOF OF LEMMA 1. Let $\{f_n\}$ be a sequence of $\mathcal{P}_1(\mathfrak{X})$ functions that converges in the Pettis norm to $f \in \mathcal{P}_1(\mathfrak{X})$. Furthermore, suppose that each f_n is essentially valued in some subspace $\mathfrak{Y}$ of $\mathfrak{X}$ and that f is essentially valued in some subspace $\mathfrak{Z}$ with $\mathfrak{Y} \subset \mathfrak{Z}$. Since f and each f_n are strongly measurable, we may assume that $\mathfrak{Z}$ is separable.

Since $\mathfrak{Z}$ is separable, there is a sequence $\{z_m^*\}$ of functionals from $\mathfrak{Z}^*$ such that an element z from $\mathfrak{Z}$ is in $\mathfrak{Y}$ if and only if $z_m^*(z) = 0$ for each m.

Since $\{f_n\}$ converges in the Pettis norm to f, for each z_m^* the sequence $\{z_m^* f_n\}_n$ converges to $z_m^* f$ in $L_1(\mathbb{R})$ -norm. But for each m, note that $z_m^* f_n$ is zero almost everywhere and so $z_m^* f$ is also zero almost everywhere. Thus f is also essentially valued in $\mathfrak{Y}$. $\square$

PROOF OF LEMMA 2. Let $\{f_n\}$ be a sequence of $\mathcal{P}_1(\mathfrak{X})$ functions which converges in Pettis norm to $f \in \mathcal{P}_1(\mathfrak{X})$. Furthermore, suppose that each f_n is essentially valued in some subspace $\mathfrak{Y}$ of $\mathfrak{X}$. Lemma 1 gives that f is also essentially valued in $\mathfrak{Y}$. Let $T : \mathfrak{Y} \to \mathfrak{X}$ be a bounded linear operator. Note that for a fixed $x^* \in \mathfrak{X}^*$ we may extend T^*x^* to an element in $\mathfrak{X}^*$ without increasing its norm.

Note that Tf (and likewise each Tf_n) is actually in $\mathcal{P}_1(\mathfrak{X})$. To see this, first observe that for a fixed $x^* \in \mathfrak{X}^*$ (viewing T^*x^* as an element in $\mathfrak{X}^*$ of the same norm)

$$\int_\Omega |x^*(Tf)|\, d\mu \;=\; \int_\Omega |(T^*x^*)(f)|\, d\mu \;\le\; ||T^*x^*||\; ||f||_{\text{Pettis}} \;.$$

and so $x^*(Tf) \in L_1$. Next, let x_E be the Pettis integral of f over $E \in \Sigma$. Note that $x_E \in \mathfrak{Y}$ and for each $x^* \in \mathfrak{X}^*$

$$x^*(Tx_E) \;=\; (T^*x^*)(x_E) \;=\; \int_E (T^*x^*)(f)\, d\mu \;=\; \int_E x^*(Tf)\, d\mu \;,$$

and so $T(x_E)$ is the Pettis integral of Tf over $E \in \Sigma$. Thus Tf is indeed in $\mathcal{P}_1(\mathfrak{X})$.

Since for $x^* \in B(X^*)$

$$\int_\Omega |x^*(Tf_n - Tf)|\, d\mu \;=\; \int_\Omega |(T^*x^*)(f_n - f)|\, d\mu \;\le\; ||T||\; ||f_n - f||_{\text{Pettis}} \;,$$

the sequence Tf_n converges in the Pettis norm to Tf. $\square$

3. VECTOR MEASURES

Let's look at our examples from a vector measure viewpoint.

A Pettis integrable function f gives rise to a vector measure $G_f\colon \Sigma \to \mathfrak{X}$ where $G_f(E)$ is the Pettis integral of f over E. The Pettis norm of f is the semivariation of G_f. If $f \in L_1(\mathfrak{X})$, then the Bochner norm of f is the variation of G_f.

Example 2 shows that for *any* infinite-dimensional Banach space $\mathfrak{X}$, there is a sequence of $\mathfrak{X}$-valued measures whose variations tend to infinity but whose semivariations tend to zero. On the other hand, Example 3 gives a $\mathfrak{X}$-valued measure of bounded semivariation but infinite variation.

For $f \in \mathcal{P}(\mathfrak{X})$, the corresponding measure G_f is μ-continuous and has a relatively compact range. In fact [DU], the completion of $\mathcal{P}_1(\mathfrak{X})$ is (isometrically) the space $K(\mathfrak{X})$ of all μ-continuous measures $G : \Sigma \to \mathfrak{X}$ whose range is relatively compact, equipped with the semivariation norm. Thus Example 4 gives a measure in $K(\mathfrak{X})$ which is not representable by a (strongly measurable) Pettis integrable function.

4. CONVERGENCE

We now restrict our attention to sequences of $L_1(\mathfrak{X})$ functions which converge (in the Bochner norm, in the Pettis norm, or weakly in $L_1(\mathfrak{X})$) to functions actually in $L_1(\mathfrak{X})$. We have the following obvious relations:

(1) Bochner-norm convergence implies weak and Pettis-norm convergence;
(2) weak convergence implies neither Bochner-norm nor Pettis-norm convergence (e.g. consider the Rademacher functions in $L_1(\mathbb{R})$);
(3) if $\mathfrak{X}$ is finite dimensional then the Pettis norm and the Bochner norm are equivalent.

Our examples help to clarify the other implications.

If $\mathfrak{X}$ is infinite-dimensional, then there is a sequence of functions which converges in Pettis norm but whose Bochner norms tend to infinity (Example 2). Thus this sequence does not converge in the Bochner norm or weakly.

What if we require the sequence to be bounded ($\sup_n ||f_n||_{L_1(\mathfrak{X})} < \infty$), or uniformly integrable, or even essentially bounded ($\sup_n ||f_n||_{L_\infty(\mathfrak{X})} < \infty$)? Again, if $\mathfrak{X}$ is infinite-dimensional then Example 1 gives a sequence of essentially bounded functions which converges in Pettis norm but not in the Bochner norm.

In certain situations we can pass from Pettis-norm convergence to weak convergence. If $f_n \to f$ in the Pettis norm, then clearly $\int_\Omega g(f_n)\,d\mu \to \int_\Omega g(f)\,d\mu$ for each *simple function* $g \in L_\infty(\mathfrak{X}^*)$. As a first step towards weak convergence, when can we at least conclude that $f_n \to f$ in the $\sigma(L_1(\mathfrak{X}), L_\infty(\mathfrak{X}^*))$-topology? A word of caution, the simple functions need not be dense in $L_\infty(\mathfrak{X}^*)$. For example, the $L_\infty(\ell_2)$ function $g(\cdot) = \sum_{n=1}^\infty e_n 1_{I_2^n}$, where $\{e_n\}$ are the standard unit vectors of ℓ_2, is at least $\frac{\sqrt{2}}{2}$ from any simple function in $L_\infty(\ell_2)$. Our next example illustrates the fact that for any infinite-dimensional Banach space there is a bounded sequence in $L_1(\mathfrak{X})$ that converges in the Pettis norm but not in the $\sigma(L_1(\mathfrak{X}), L_\infty(\mathfrak{X}^*))$-topology.

EXAMPLE 5.

Consider the sequence $\{g_n\}$ from Example 3 where

$$g_n(\omega) \;=\; 2^n \; f_n(2^n\omega - 1) \; 1_{I_2^n}(\omega) \; .$$

Since $||f_n||_{L_1(\mathfrak{X})} \leq 2$, the Bochner norm of each g_n is at most 2.

To examine the Pettis norm, fix $x^* \in B(X^*)$. Computations in Example 1 show that

$$\int_\Omega |x^* g_n|\,d\mu \;=\; \int_{I_2^n} |x^* g_n|\,d\mu \;=\; \int_\Omega |x^* f_n|\,d\mu \;\leq\; 2\; 2^{-\frac{n}{2}}$$

and so

$$\|g_n\|_{\text{Pettis}} \;\le\; 2\, 2^{-\frac{n}{2}} \; .$$

Thus $g_n \to 0$ in the Pettis norm.

As for the $\sigma(L_1(\mathfrak{X}), L_\infty(\mathfrak{X}^*))$-topology, find $y_k^n \in B(\mathfrak{X}^*)$ such that $y_k^n(e_k^n) = \|e_k^n\|_{\mathfrak{X}}$. Note that $\{I_2^n : n = 1, 2, \dots\}$ partitions $[0,1]$ and that $\{I_j^{2n} : j = 2^n + 1, \dots, 2^{n+1}\}$ partitions the interval I_2^n. Define $g\colon \Omega \to \mathfrak{X}^*$ by

$$g(\omega) \;=\; \sum_{n=1}^{\infty} \sum_{k=1}^{2^n} y_k^n 1_{I^{2n}_{2^n+k}}(\omega) \; .$$

Clearly $g \in L_\infty(\mathfrak{X}^*)$. But

$$\begin{aligned} \int_\Omega g(g_n)\, d\mu \;&=\; \int_{I_k^n} g(g_n)\, d\mu \;=\; \sum_{k=1}^{2^n} \int_{I^{2n}_{2^n+k}} g(g_n)\, d\mu \\ &=\; \sum_{k=1}^{2^n} \int_{I^{2n}_{2^n+k}} y_k^n(2^n\; e_k^n)\, d\mu \;\ge\; \sum_{k=1}^{2^n} 2^n\; 2^{-2n}\; 1 \;=\; 1 \; . \end{aligned}$$

Thus $\{g_n\}$ does not converge in the $\sigma(L_1(\mathfrak{X}), L_\infty(\mathfrak{X}^*))$-topology. $\square$

However, a *uniformly integrable* Pettis-norm convergent sequence does converge in the $\sigma(L_1(\mathfrak{X}), L_\infty(\mathfrak{X}^*))$-topology. To see why, consider a uniformly integrable sequence $\{f_n\}$ such that $\int_\Omega g(f_n)\, d\mu \to 0$ for each simple function $g \in L_\infty(\mathfrak{X}^*)$. For an arbitrary $g \in L_\infty(\mathfrak{X}^*)$, note that there is a sequence of simple functions which converges to g in measure. So we can approximate g (in $L_\infty(\mathfrak{X}^*)$ norm) by a simple function $\tilde{g}$ on a subset $E \in \Sigma$ of full enough measure that $\int_{E^c} \|f_n\|\, d\mu$ is small for each n and the measure of E^c is also small. Note that

$$\left| \int_\Omega g(f_n)\, d\mu \right| \;\le\; \int_{E^c} \|g\|_{\mathfrak{X}^*} \|f_n\|_{\mathfrak{X}}\, d\mu + \int_E \|g - \tilde{g}\|_{\mathfrak{X}^*} \|f_n\|_{\mathfrak{X}}\, d\mu + \left| \int_E \tilde{g}(f_n)\, d\mu \right| \; .$$

The first two integrals on the left-hand side are small for each n. The last integral converges to zero as $n \to \infty$. Thus $\int_\Omega g(f_n)\, d\mu \to 0$, as needed.

A Pettis-norm convergent sequence need not be uniformly integrable (Example 2). However, if a sequence converges in the $\sigma(L_1(\mathfrak{X}), L_\infty(\mathfrak{X}^*))$-topology then it is *necessarily* uniformly integrable. To see why this is so, recall the usual proof [DU p. 104] that a weakly convergent sequence is uniformly integrable. If a subset K of $L_1(\mathfrak{X})$ is not uniformly integrable, then we can find (with the help of Rosenthal's Lemma) a sequence $\{f_n\}$ in K and a disjoint sequence $\{E_n\}$ in Σ such that

$$\int_{E_n} \|f_n\|\, d\mu \;>\; 2\delta \qquad \text{and} \qquad \int_{\cup_{j \ne n} E_j} \|f_n\|\, d\mu \;<\; \delta$$

for some $\delta > 0$. For each n we can find $l_n \in L_\infty(\mathfrak{X}^*) \subset [L_1(\mathfrak{X})]^*$ of norm one and supported on E_n such that $\int_{E_n} l_n(f_n) > 2\delta$. Define $l\colon \Omega \to \mathfrak{X}^*$ by

$$l(\cdot) \;=\; \sum_{n=1}^{\infty} l_n(\cdot)\, 1_{E_n}(\cdot)\ .$$

Clearly, l is a norm one element of $L_\infty(\mathfrak{X}^*) \subset [L_1(\mathfrak{X})]^*$. But $\int_\Omega l(f_n) \nrightarrow 0$ since

$$\int_\Omega l(f_n)\, d\mu \;=\; \int_{E_n} l_n(f_n)\, d\mu \;+\; \int_{F_n} l(f_n)\, d\mu\ ,$$

where $F_n \;=\; \cup_{m\neq n} E_m$, and so

$$\begin{aligned}
\left|\int_\Omega l(f_n)\, d\mu\right| \;&\geq\; \left|\int_{E_n} l_n(f_n)\, d\mu\right| \;-\; \int_{F_n} \|l\|_{\mathfrak{X}^*}\, \|(f_n)\|_{\mathfrak{X}}\, d\mu \\
&\geq\; 2\delta \;-\; \int_{F_n} \|f_n\|\, d\mu \\
&\geq\; 2\delta \;-\; \delta \;=\; \delta\ .
\end{aligned}$$

So if $g_n \to g$ in the $\sigma(L_1(\mathfrak{X}), L_\infty(\mathfrak{X}^*))$-topology then the set $\{g_n - g\}$, and thus also the set $\{g_n\}$, are uniformly integrable.

To pass from $\sigma(L_1(\mathfrak{X}), L_\infty(\mathfrak{X}^*))$-convergence to weak convergence, recall [DU] that the dual of $L_1(\mathfrak{X})$ is (isometrically) $L_\infty(\mathfrak{X}^*)$ if and only if $\mathfrak{X}^*$ has the Radon-Nikodým property (RNP). Thus we have:

FACT. *If $\mathfrak{X}^*$ has the RNP, then a uniformly integrable Pettis-norm convergent sequence of $L_1(\mathfrak{X})$ functions also converges weakly.*

As noted above, the uniform integrability condition is necessary. Our next proposition shows that it is also necessary that $\mathfrak{X}^*$ have the RNP. The proof, which uses Stegall's Factorization Theorem, may be pulled out of a result of Ghoussoub and P. Saab [GS] characterizing weakly compact sets in $L_1(\mathfrak{X})$.

PROPOSITION. *If $\mathfrak{X}^*$ fails the RNP, then there is an essentially bounded sequence of $L_1(\mathfrak{X})$ functions that is Pettis-norm convergent but not weakly convergent.*

PROOF.

Since $\mathfrak{X}^*$ fails the RNP, there is a separable subspace $\mathfrak{Y}$ of $\mathfrak{X}$ such that $\mathfrak{Y}^*$ is not separable. We shall construct an essentially bounded sequence $\{g_n\}$ of $L_1(\mathfrak{Y})$ functions such that $g_n \to 0$ in the Pettis-norm on $L_1(\mathfrak{Y})$ but $g_n \nrightarrow 0$ weakly in $L_1(\mathfrak{Y})$. This is sufficient.

Let $\Delta = \{-1, 1\}^{\mathbb{N}}$ be the Cantor group with Haar measure ν. Let $\{\Delta^n_k\colon k = 1, \ldots, 2^n\}$ be the standard n-th partition of Δ. Thus $\Delta^0_1 = \Delta$ and $\Delta^n_k = \Delta^{n+1}_{2k-1} \cup \Delta^{n+1}_{2k}$ and $\nu(\Delta^n_k) = 2^{-n}$. We will work with our underlying measure space being the Cantor group endowed with its Haar measure instead of our usual Lebesgue measure space on $[0, 1]$.

Consider the set $C(\Delta)$ of real-valued continuous function on Δ as a subspace of $L_\infty(\mathbb{R})$. Let $\{1_\Delta\} \cup \{h_k^n \colon n = 0, 1, 2, \ldots \text{ and } k = 1, \ldots, 2^n\}$ be the usual Haar basis of $C(\Delta)$, where $h_k^n \colon \Delta \to \mathbb{R}$ is given by

$$h_k^n \;=\; 1_{\Delta_{2k-1}^{n+1}} \;-\; 1_{\Delta_{2k}^{n+1}} \; .$$

Let $\{e_k^n \colon n = 0, 1, 2, \ldots \text{ and } k = 1, \ldots, 2^n\}$ be an enumeration (lexicographically) of the usual ℓ_1 basis and $H \colon \ell_1 \to L_\infty$ be the Haar operator that takes e_k^n to h_k^n.

By Stegall's Factorization Theorem [S, Theorem 4], H factors through $\mathfrak{Y}$, i.e. there are bounded linear operators $R \colon \ell_1 \to \mathfrak{Y}$ and $S \colon \mathfrak{Y} \to L_\infty$ such that $H = SR$. Let $\tilde{R}$ be the natural extension of R to a bounded linear operator from $L_1(\ell_1)$ to $L_1(\mathfrak{Y})$.

Consider the sequence $\{f_m\}$ of $L_1(\ell_1)$ functions given by

$$f_m(\cdot) \;=\; \frac{1}{m} \sum_{n=1}^{m} \sum_{k=1}^{2^n} h_k^n(\cdot) e_k^n \; .$$

Let $g_m = \tilde{R}(f_m)$. The sequence $\{g_m\}$ of $L_1(\mathfrak{Y})$ functions does the job.

Since $\sup_m \|f_m(\omega)\|_{\ell_1} = 1$ for ν-a.e. ω, we have that $\{g_m\}$ is essentially bounded.

To examine the Pettis norm of f_m, consider the continuous convex function $\psi \colon \ell_\infty \to \mathbb{R}$ where the image of $x^* = (\alpha_k^n) \in \ell_\infty \simeq \ell_1^*$ (again, lexicographically ordered) is

$$\psi(x^*) \;=\; \int_\Delta |x^*(f_m)| \, d\nu \;=\; \frac{1}{m} \int_\Delta \left| \sum_{n=1}^{m} \sum_{k=1}^{2^n} h_k^n(\cdot) x^*(e_k^n) \right| d\nu \; .$$

Since the image of f_m is supported on $\{e_k^n \colon n = 1, \ldots, 2^m \text{ and } k = 1, \ldots, 2^n\}$, we consider the natural restriction map $r_m \colon \ell_\infty \to \ell_\infty^{2(2^{2^m}-1)} \subset \ell_\infty$ given by

$$\begin{aligned} r_m((\alpha_k^n \colon n = 0, 1, 2, \ldots \text{ and } k = 1, \ldots, 2^n)) \;=\; \\ (\alpha_k^n \colon n = 1, \ldots, 2^m \text{ and } k = 1, \ldots, 2^n) \; . \end{aligned}$$

Let K be the unit ball of $r_m(\ell_\infty)$. The maximum of ψ over K is attained at some extreme point x_0^* of K. Being an extreme point, $x_0^* = (\epsilon_k^n)$ satisfies $|\epsilon_k^n| = 1$ for each admissible n and k. So

$$\begin{aligned} \|f_m\|_{\text{Pettis}} \;&=\; \sup_{x^* \in B(\ell_\infty)} \psi(x^*) \;=\; \sup_{x^* \in K} \psi(x^*) \\ &=\; \psi(x_0^*) \;=\; \frac{1}{m} \int_\Delta \left| \sum_{n=1}^{m} \sum_{k=1}^{2^n} \epsilon_k^n \; h_k^n(\cdot) \right| d\nu \; . \end{aligned}$$

Since $\sum_{k=1}^{2^n} \epsilon_k^n \; h_k^n$ behaves like the n^{th}-Rademacher function, Khintchine's inequality gives that $\|f_m\|_{\text{Pettis}}$ decays like $m^{-\frac{1}{2}}$.

Now for the Pettis norm of g_m, note that if $y^* \in \mathfrak{Y}^*$ then

$$\begin{aligned}\int_\Delta |y^* g_m| \, d\nu &= \int_\Delta |y^*(R f_m)| \, d\nu \\ &= \int_\Delta |(R^* \, y^*)(f_m)| \, d\nu \ \leq \ ||R^*|| \, ||y^*|| \, ||f_m||_{\text{Pettis}} \ .\end{aligned}$$

Thus $g_m \to 0$ in the Pettis norm on $L_1(\mathfrak{Y})$.

To see that $g_m \nrightarrow 0$ weakly in $L_1(\mathfrak{Y})$, it suffices to show that $\tilde{H} f_m \nrightarrow 0$ weakly in $L_1(L_\infty)$ where $\tilde{H}$ is the natural extension of H to an operator from $L_1(\ell_1)$ to $L_1(L_\infty)$. But since

$$\tilde{H} f_m(\cdot) \ = \ \frac{1}{m} \sum_{n=1}^{m} \sum_{k=1}^{2^n} h_k^n(\cdot) h_k^n \ ,$$

we may view $\tilde{H} f_m$ as a function from Δ into $C(\Delta)$ and thus we only need to show that $\tilde{H} f_m \nrightarrow 0$ weakly in $L_1(C(\Delta))$. To see this, just consider the linear functional $l \in [L_1(C(\Delta))]^*$ given by

$$l(f) \ = \ \int_\Delta [f(\omega)](\omega) \, d\nu \qquad \text{where} \qquad f \in L_1(C(\Delta))$$

and note that

$$l(\tilde{H} f_m) \ = \ \frac{1}{m} \int_\Delta \sum_{n=1}^{m} \sum_{k=1}^{2^n} h_k^n(\omega) h_k^n(\omega) \, d\nu \ = \ 1 \ .$$

Thus the sequence $\{g_m\}$ does all it is supposed to do. $\square$

We close with one last example illustrating the above proposition in the case that $\mathfrak{X}$ is ℓ_1.

EXAMPLE 6. Once again working on the Lebesgue measure space on $[0,1]$, consider the sequence $\{g_n\}$ of $L_1(\ell_1)$ functions where

$$g_n(\cdot) \ = \ \frac{1}{n} \sum_{k=1}^{n} r_k(\cdot) e_k$$

where $\{e_k\}$ are the standard unit vectors of ℓ_1 and $\{r_k\}$ are the Rademacher functions.

The sequence is uniformly integrable, indeed it is even essentially bounded by 1. As for the Pettis norm of a g_n, fix $x^* \in (\ell_1)^*$, say $x^* = (\alpha_j) \in \ell_\infty$. Since

$$\int_\Omega |x^* g_n| \, d\mu \ = \ \frac{1}{n} \int_\Omega |\sum_{k=1}^{n} \alpha_k r_k| \, d\mu \ ,$$

Khintchine's inequality shows us that the Pettis norm of g_n behaves like $\frac{1}{\sqrt{n}}$. Thus $g_n \to 0$ in the Pettis norm and so also in the $\sigma(L_1(\mathfrak{X}), L_\infty(\mathfrak{X}^*))$-topology.

However, $\{g_n\}$ does not converge weakly. For consider the vector measure $G \colon \Sigma \to \ell_\infty$ given by

$$G(E) \ = \ (\mu(E \cap [r_j = 1])) \ .$$

Clearly G is a μ-continuous vector measure of bounded variation. Thus G gives rise to a functional $l \in [L_1(\ell_1)]^*$ given by

$$l(f) \;=\; \int_\Omega f\,dG \qquad \text{for} \quad f \in L_1(\ell_1)\ ,$$

where we view G as taking values in ℓ_1^*. Since

$$l(g_n) \;=\; \frac{1}{n}\sum_{k=1}^{n} l(e_k r_k) \;=\; \frac{1}{n}\sum_{k=1}^{n} l(e_k 1_{[r_k=1]} \,-\, e_k 1_{[r_k=-1]}) \;=\; \frac{1}{n}\sum_{k=1}^{n} \tfrac{1}{2} - 0 \;=\; \frac{1}{2}\ ,$$

we see that $\{g_n\}$ does not converge weakly.

Note that the above linear functional $l\colon L_1(\ell_1) \to \mathbb{R}$ is continuous for the Bochner norm but not for the Pettis norm. $\square$

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