WEAK VS. NORM COMPACTNESS
IN $L_1$: THE BOCCE CRITERION

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Abstrac t. We present a new simple proof that if a relatively weakly compact subset of $L_1$ satisfies the Bocce criterion (an oscillation condition), then it is relatively norm compact. The converse of this fact is easy to verify. A direct consequence is that, for a bounded linear operator $T$ from $L_1$ into a Banach space $\mathfrak{X}$, $T$ is Dunford-Pettis if and only if the subset $T^*(B(\mathfrak{X}^*))$ of $L_1$ satisfies the Bocce criterion.

A relatively weakly compact subset of $L_1$ is relatively norm compact if and only if it satisfies the Bocce criterion (an oscillation condition) [G1]. We shall present a new simple proof that if a relatively weakly compact subset of $L_1$ satisfies the Bocce criterion, then it is relatively norm compact. The converse is easy to verify.

Recall that a Banach space $\mathfrak{X}$ has the complete continuity property (CCP) if each bounded linear operators from $L_1$ into $\mathfrak{X}$ is Dunford-Pettis (i.e. maps weakly convergent sequences onto norm convergent sequences). The CCP is a weakening of the Radon-Nikodým property and of strong regularity. Since a bounded linear operator $T$ from $L_1$ into $\mathfrak{X}$ is Dunford-Pettis if and only if the subset $T^*(B(\mathfrak{X}^*))$ of $L_1$ is relatively norm compact, the above fact gives that $T$ is Dunford-Pettis if and only if $T^*(B(\mathfrak{X}^*))$ satisfies the Bocce criterion. This oscillation characterization of Dunford-Pettis operators leads to dentability and tree characterizations of the CCP [G2]. Namely, $\mathfrak{X}$ has the CCP if and only if all bounded subsets of $\mathfrak{X}$ are weak-norm-one dentable. Also, $\mathfrak{X}$ has the CCP if and only if no bounded separated $\delta$-trees grow in $\mathfrak{X}$, or equivalently, no bounded $\delta$-Rademacher trees grow in $\mathfrak{X}$.

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Throughout this note, $X$ denotes an arbitrary Banach space. The triple $(\Omega, \Sigma, \mu)$ refers to the Lebesgue measure space on $[0, 1]$, $\Sigma^+$ to the sets in $\Sigma$ with positive measure, and $L_1$ to $L_1(\Omega, \Sigma, \mu)$. All notation and terminology, not otherwise explained, are as in [DU].

[G1] introduces the following definitions.

*Definitions.* For $f$ in $L_1$ and $A$ in $\Sigma^+$, the Bocce oscillation of $f$ on $A$ is given by

$$\text{Bocce-osc } f\big|_A \equiv \frac{\int_A \left| f - \frac{\int_A f \, d\mu}{\mu(A)} \right| \, d\mu}{\mu(A)}.$$  

A subset $K$ of $L_1$ satisfies the *Bocce criterion* if for each $\epsilon > 0$ and $B$ in $\Sigma^+$ there is a finite collection $\mathcal{F}$ of subset of $B$ each with positive measure such that for each $f$ in $K$ there is an $A$ in $\mathcal{F}$ satisfying $\text{Bocce-osc } f\big|_A < \epsilon$.

This note’s main purpose is to present a new proof to the theorem below. The author is grateful to Michel Talagrand for his helpful discussions concerning this theorem and proof.

**Theorem.** If a relatively weakly compact subset of $L_1$ satisfies the Bocce criterion, then it is relatively $L_1$-norm compact.

We need the following lemma which we shall verify after the proof of the Theorem.

**Lemma.** If a subset of $L_1$ satisfies the Bocce criterion, then the translate of that set by a $L_1$-function also satisfies the Bocce criterion.

**Proof of the Theorem.** Assume that the relatively weakly compact subset $K$ of $L_1$ is not relatively norm compact. We shall show that $K$ does not satisfy the Bocce criterion.
Since $K$ is not relatively norm compact but is relatively weakly compact, there is a sequence \( \{f_n\} \) in a translate $\tilde{K}$ of $K$ satisfying

1. $\{f_n\}$ has no $L_1$-convergent subsequence
2. $\{f_n\}$ converges weakly in $L_1$ to 0
3. \( \{ |f_n| \} \) converges weakly in $L_1$, say to $f$
4. $\int f \, d\mu \geq 4\epsilon$ for some $\epsilon > 0$.

Set $B = [f \geq 3\epsilon]$. Condition (4) guarantees that $B \in \Sigma^+$.

Let $\mathcal{F}$ be a finite collection of subsets of $B$, each with positive measure. Choose $N$ such that for each $A \in \mathcal{F}$

5. $|\int_A f_N \, d\mu| < \epsilon \mu(A)$ (possible by (2))
6. $|\int_A f \, d\mu - \int_A |f_N| \, d\mu| < \epsilon \mu(A)$ (possible by (3)).

Then for each $A \in \mathcal{F}$ we have that

\[
\text{Bocce-osc } f_N|_A \equiv \frac{\int_A |f_N - \frac{\int_A f_N \, d\mu}{\mu(A)}| \, d\mu}{\mu(A)} \geq \frac{\int_A |f_N| \, d\mu}{\mu(A)} - \frac{\int_A f_N \, d\mu}{\mu(A)} \\
\geq \frac{\int_A f \, d\mu - \epsilon \mu(A)}{\mu(A)} \geq \frac{3\epsilon \mu(A)}{\mu(A)} - \epsilon = \epsilon.
\]

Thus, $\tilde{K}$ does not satisfy the Bocce criterion and so $K$ also does not satisfy the Bocce criterion.

Proof of the Lemma. Let the subset $K$ of $L_1$ satisfies the Bocce criterion and $f \in L_1$. We need to show that the set $K+f \equiv \{ g+f : g \in K \}$ satisfies the Bocce criterion. Towards this end, fix $\epsilon > 0$ and $B \in \Sigma^+$. Find $B_0 \subset B$ with $B_0 \in \Sigma^+$ such that $f$ is bounded on $B_0$.

Approximate $f\chi_{B_0}$ in $L_\infty$-norm within $\frac{\epsilon}{4}$ by a simple function $\tilde{f}$. Find $C \subset B_0$ with $C \in \Sigma^+$ such that $\tilde{f}$ is constant on $C$. Since $K$ satisfies the Bocce criterion, we can find a finite collection $\mathcal{F}$ of subsets corresponding to $\frac{\epsilon}{2}$ and $C$. 


Fix \( g + f \in K + f \). Find \( A \in \mathcal{F} \) such that \( \text{Bocce-osc} \ g \big|_A < \frac{\varepsilon}{2} \). Note that since \( \tilde{f} \) is constant on \( A \), \( \text{Bocce-osc} \ g \big|_A = \text{Bocce-osc} \ (g + \tilde{f}) \big|_A \). Now,

\[
\text{Bocce-osc} \ (g + f) \big|_A \leq \text{Bocce-osc} \ (g + \tilde{f}) \big|_A + \text{Bocce-osc} \ (\tilde{f} - f) \big|_A \leq \text{Bocce-osc} \ g \big|_A + 2 \| (\tilde{f} - f) \chi_A \|_{L_\infty} < \varepsilon.
\]

Thus \( K + f \) satisfies the Bocce criterion. \( \blacksquare \)

REFERENCES

[G2]. Maria Girardi, Dentability, Trees, and Dunford-Pettis operators on \( L_1 \), (to appear in Pacific J. Math.).

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