DUNFORD-PETTIS OPERATORS ON L_1 AND THE COMPLETE CONTINUITY PROPERTY

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ABSTRACT

The interplay between the behavior of bounded linear operators from L_1 into a Banach space $\mathfrak{X}$ and the internal geometry of $\mathfrak{X}$ has long been evident. The Radon-Nikodým property (RNP) and strong regularity arose as operator theoretic properties but were later realized as geometric properties.

Another operator theoretic property, the complete continuity property (CCP), is a weakening of both the RNP and strong regularity. A Banach space $\mathfrak{X}$ has the CCP if all bounded linear operators from L_1 into $\mathfrak{X}$ are Dunford-Pettis (i.e. take weakly convergent sequences to norm convergent sequences). There are motivating partial results suggesting that the CCP also can be realized as a geometric property. This thesis provides such a realization.

Our first step is to derive an oscillation characterization of Dunford-Pettis operators. Using this oscillation characterization, we obtain a geometric description of the CCP; namely, we show that $\mathfrak{X}$ has the CCP if and only if all bounded subsets of $\mathfrak{X}$ are Bocce dentable, or equivalently, all bounded subsets of $\mathfrak{X}$ are weak-norm-one dentable. This geometric description leads to yet another; $\mathfrak{X}$ has the CCP if and only if no bounded separated δ-trees grow in $\mathfrak{X}$, or equivalently, no bounded δ-Rademacher trees grow in $\mathfrak{X}$. We also localize these results. We motivate these characterizations by the corresponding (known) characterizations of the RNP and of strong regularity.

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CHAPTER 1

INTRODUCTION

The interplay between the behavior of bounded linear operators from L_1 into a Banach space $\mathfrak{X}$ and the internal geometry of $\mathfrak{X}$ has long been evident. The Radon-Nikodým property (RNP) was originally defined in terms of the behavior of operators from L_1 into $\mathfrak{X}$. Rieffel, Maynard, Huff, Davis, and Phelps provided the first concrete realization of the RNP as a geometric property of a Banach space when they showed that $\mathfrak{X}$ has the RNP if and only if all bounded subsets of $\mathfrak{X}$ are dentable. From this geometric characterization follows another; namely, $\mathfrak{X}$ has the RNP if and only if no bounded δ-bushes grow in $\mathfrak{X}$, and in the case that $\mathfrak{X}$ is a dual space (Stegall), if and only if no bounded δ-trees grow in $\mathfrak{X}$.

Similarly, the concept of strong regularity, a weakening of the RNP, arose as an operator theoretic property [B1] but was later realized as a geometric property [GGMS].

The complete continuity property (CCP), another operator theoretic property, has been around for many years. Dunford and Pettis showed that a Banach space with the RNP has the CCP; in fact, a strongly regular space also has the CCP.

There are motivating partial results suggesting that the CCP also can be realized as a geometric property. Bourgain showed that $\mathfrak{X}$ has the CCP if no bounded δ-trees grow in $\mathfrak{X}$. As for the case in which $\mathfrak{X}$ is a dual space, work of Pelczynski, Hagler, and Bourgain gives that $\mathfrak{X}$ has the CCP if and only if $\mathfrak{X}$ is strongly regular.

In this thesis, we shall complete the cycle by showing that the complete continuity property is indeed an internal geometric property of a Banach space.

Our first step is to derive an oscillation characterization of Dunford-Pettis operators (Chapter 2). Using this oscillation characterization, we obtain a geometric description of the CCP; namely, we show that $\mathfrak{X}$ has the CCP if and only if all bounded subsets of $\mathfrak{X}$ are Bocce dentable (Chapter 3). This geometric description leads to yet another; $\mathfrak{X}$ has the CCP if and only if no bounded separated δ-trees grow in $\mathfrak{X}$ (Chapter 4). We motivate these characterizations by the corresponding (known) characterizations of the Radon-Nikodým property and of strong regularity.

Throughout this thesis, $\mathfrak{X}$ denotes an arbitrary Banach space, $\mathfrak{X}^*$ the dual space of $\mathfrak{X}$, $B(\mathfrak{X})$ the closed unit ball of $\mathfrak{X}$, and $S(\mathfrak{X})$ the unit sphere of $\mathfrak{X}$. The triple (Ω, Σ, μ) refers to the Lebesgue measure space on $[0,1]$, Σ^+ to the sets in Σ with positive measure, L_1 to $L_1(\Omega, \Sigma, \mu)$, and L_1-compact to compact in the L_1-norm topology. A subset K of L_1 is uniformly bounded if $\sup_{f \in K} \| f \|_{L_\infty}$ is finite. All notation and terminology, not otherwise explained, are as in [DU]. For clarity, known results are presented as FACTS while new results are presented as THEOREMS, COROLLARIES, LEMMAS, and OBSERVATIONS.

As for basic definitions, the RNP, strong regularity, and the CCP each have several equivalent formulations. We shall define these properties by examining the behavior of bounded linear operators from L_1 into $\mathfrak{X}$.

Recall that a bounded linear operator $T : L_1 \to \mathfrak{X}$ is <u>(Bochner) representable</u> if there is a g in $L_\infty(\mu, \mathfrak{X})$ such that $Tf = \int_\Omega fg \, d\mu$ for each f in $L_1(\mu)$. A Banach space $\mathfrak{X}$ has the <u>Radon-Nikodým property</u> if all bounded linear operators from L_1 into $\mathfrak{X}$ are Bochner representable.

A bounded linear operator $T : L_1 \to \mathfrak{X}$ is <u>strongly regular</u> if for each bounded subset D of L_1 and $\epsilon > 0$ there are finitely many slices $S_1, S_2, \ldots, S_n$ of D satisfying

$$\operatorname{diam}\left(\frac{TS_1 + \ldots + TS_n}{n}\right) < \epsilon ,$$

where a slice of a subset D of Banach space $\mathfrak{Y}$ is any non-empty set of the form $\{y \in D\colon \ y^*(y) > \alpha\}$ where $y^* \in \mathfrak{Y}^*$ and $\alpha \in \mathbb{R}$ are fixed. A Banach space $\mathfrak{X}$ is strongly regular if all bounded linear operators from L_1 into $\mathfrak{X}$ are strongly regular.

An operator is Dunford-Pettis if it maps weakly convergent sequences to norm convergent sequences. A Banach space $\mathfrak{X}$ has the complete continuity property (CCP) if all bounded linear operators from L_1 into $\mathfrak{X}$ are Dunford-Pettis.

It is well known that if a bounded linear operator T from L_1 into $\mathfrak{X}$ is representable then T is strongly regular; and if T is strongly regular then T is Dunford-Pettis. Thus, if $\mathfrak{X}$ has the Radon-Nikodým property then $\mathfrak{X}$ is strongly regular, and if $\mathfrak{X}$ is strongly regular then $\mathfrak{X}$ has the CCP. The dual of the James tree space [cf. J] is strongly regular yet fails the RNP. Talagrand [cf. T1] has constructed a space that has the CCP yet is not strongly regular.

The following fact provides several equivalent formulations of the CCP.

Fact 1.1. For a bounded linear operator T from L_1 into $\mathfrak{X}$, the following statements are equivalent.

(1) T is a Dunford-Pettis operator.

(2) T maps weak compact sets to norm compact sets.

(3) $T(B(L_\infty))$ is a relatively norm compact subset of $\mathfrak{X}$.

(4) The corresponding vector measure $F : \Sigma \to \mathfrak{X}$ given by $F(E) = T(\chi_E)$ has a relatively norm compact range in $\mathfrak{X}$.

(5) The adjoint of the restriction of T to L_∞ from $\mathfrak{X}^*$ into L_∞^* is a compact operator.

(6) As a subset of L_1, $T^*(B(\mathfrak{X}^*))$ is relatively L_1-compact.

(7) $E_\pi T^*$ converges to T^* in the operator topology of $\mathcal{L}(\mathfrak{X}^*, L_1)$.

(8) TE_π converges to T in the operator topology of $\mathcal{L}(L_\infty, \mathfrak{X})$.

The equivalence of (2) and (3) follows from the fact that the subsets of L_1 that are relatively weakly compact are precisely those subsets that are bounded and uniformly integrable, which in turn, are precisely those subsets that can be uniformly approximated in L_1-norm by uniformly bounded subsets. The other implications in Fact 1.1 are straightforward and easy to verify. Because of (4), the CCP is also referred to as the compact range property (CRP).

Towards a martingale characterization of the CCP, fix an increasing sequence $\{\pi_n\}_{n\geq 0}$ of finite positive interval partitions of Ω such that $\vee\ \sigma(\pi_n) = \Sigma$ and $\pi_0 = \{\Omega\}$. Let $\mathcal{F}_n$ denote the sub-σ-field $\sigma(\pi_n)$ of Σ that is generated by π_n. For f in $L_1(\mathfrak{X})$, let $E_n(f)$ denote the conditional expectation of f given $\mathcal{F}_n$.

Definition 1.2. A sequence $\{f_n\}_{n\geq 0}$ in $L_1(\mathfrak{X})$ is an $\mathfrak{X}$-valued <u>martingale</u> with respect to $\{\mathcal{F}_n\}$ if for each n we have that f_n is $\mathcal{F}_n$-measurable and $E_n(f_{n+1}) = f_n$. The martingale $\{f_n\}$ is <u>uniformly bounded</u> provided that $\sup_n \| f_n \|_{L_\infty}$ is finite. Often the martingale is denoted by $\{f_n, \mathcal{F}_n\}$ in order to display both the functions and the sub-σ-fields involved.

There is a one-to-one correspondence between the bounded linear operators T from L_1 into $\mathfrak{X}$ and the uniformly bounded $\mathfrak{X}$-valued martingales $\{f_n, \mathcal{F}_n\}$. This correspondence is obtained by taking

$$T(g) = \lim_{n\to\infty} \int_\Omega f_n(\omega) g(\omega)\, d\mu(\omega) \qquad \text{if } \{f_n\} \text{ is the martingale,}$$

and

$$f_n(\omega) = \sum_{E\in\pi_n} \frac{T(\chi_E)}{\mu(E)} \chi_E(\omega) \qquad \text{if } T \text{ is the operator.}$$

It is easy to verify that a bounded linear operator from L_1 into $\mathfrak{X}$ is representable by $f \in L_\infty(\mu, \mathfrak{X})$ if and only if the corresponding martingale converges in $L_1(\mathfrak{X})$-norm to f. Fact 1.1.6 implies that a bounded linear operator T from L_1 into $\mathfrak{X}$ is

Dunford-Pettis if and only if

$$\lim_{m,n\to\infty} \sup_{x^*\in B(\mathfrak{X}^*)} \| E_n(T^*x^*) - E_m(T^*x^*) \|_{L_1} = 0 \ .$$

Since $E_n(T^*x^*) = x^*f_n$ in L_1, we have the following martingale characterization of Dunford-Pettis operators, and thus of the CCP.

Fact 1.3. A bounded linear operator from L_1 into $\mathfrak{X}$ is Dunford-Pettis if and only if the corresponding martingale is Cauchy in the Pettis norm. Consequently, a Banach space $\mathfrak{X}$ has the CCP if and only if all uniformly bounded $\mathfrak{X}$-valued martingales are Pettis-Cauchy.

CHAPTER 2

OSCILLATION

Fix a bounded linear operator T from L_1 into $\mathfrak{X}$ and consider the uniformly bounded subset $T^*(B(\mathfrak{X}^*)) = \{ T^*(x^*) \ : \| x^* \| \leq 1\}$ of L_1. In this chapter, we shall explore the oscillation behavior of elements in $T^*(B(\mathfrak{X}^*))$ in order to attain information about T, and thus also geometric information about $\mathfrak{X}$.

We can determine if T is representable by examining $T^*(B(\mathfrak{X}^*))$ [cf. GGMS].

Fact 2.1. The following statements are equivalent.

(1) T is representable.

(2) $T^*(B(\mathfrak{X}^*))$ is equi-measurable, i.e.

for each $\epsilon > 0$ there is an A in Σ^+ with $\mu(A) > 1-\epsilon$ such that, as a subset of L_∞, $\{(T^*x^*)\big|_A \ : \| x^* \| \leq 1\}$ is relatively L_∞-norm-compact.

(3) For each $\epsilon > 0$ and B in Σ^+ there is a subset A of B with positive measure satisfying

$$\operatorname{diam} \left\{ \frac{T(\chi_E)}{\mu(E)} \ : \ E \subset A \text{ and } E \in \Sigma^+ \right\} < \epsilon \ .$$

Recall that for a bounded function f in L_1 and A in Σ^+, the <u>oscillation of f restricted to A</u> is given by

$$\operatorname{osc} f\big|_A \ = \ \operatorname{ess} \sup_{\omega \in A} f(\omega) \ - \ \operatorname{ess} \inf_{\omega \in A} f(\omega) \ .$$

Condition (3) may be viewed as an oscillation condition on $T^*(B(\mathfrak{X}^*))$ since

$$\operatorname{diam} \left\{ \frac{T(\chi_E)}{\mu(E)} \ : \ E \subset A \text{ and } E \in \Sigma^+ \right\} \quad = \quad \sup_{x^* \in B(\mathfrak{X}^*)} \operatorname{osc} \, (T^*x^*)\big|_A \ .$$

We can also determine whether T is strongly regular by examining the oscillation of elements in $T^*(B(\mathfrak{X}^*))$. Towards this, recall the following well-known fact [cf. GGMS].

Fact 2.2. If K is a uniformly bounded subset of L_1, then the following statements are equivalent.

(1) K is a <u>set of small oscillation</u>, i.e.

for each $\epsilon > 0$ there is a finite positive measurable partition $\mathcal{P}$ of Ω such that for each f in K

$$\sum_{A \in \mathcal{P}} \mu(A) \text{ osc } f\big|_A \; < \; \epsilon \; .$$

(2) K has the <u>Bourgain property</u>, i.e.

for each $\epsilon > 0$ and B in Σ^+ there is a finite collection $\mathcal{F}$ of subsets of B each with positive measure such that for each f in K there exists an A in $\mathcal{F}$ satisfying

$$\text{osc } f\big|_A < \epsilon \; .$$

Applying this fact to the uniformly bounded subset $T^*(B(\mathfrak{X}^*))$ provides the following characterization of strongly regular operators [cf. GGMS].

Fact 2.3. The following statements are equivalent.

(1) T is a strongly regular operator.

(2) $T^*(B(\mathfrak{X}^*))$ is a set of small oscillation.

(3) $T^*(B(\mathfrak{X}^*))$ has the Bourgain property .

Clearly, the oscillation condition [Fact 2.1.3] on $T^*(B(\mathfrak{X}^*))$ that characterizes representable operators implies the oscillation condition [Fact 2.3.3] on $T^*(B(\mathfrak{X}^*))$ that characterizes strongly regular operators. The natural question to explore next is what oscillation condition on $T^*(B(\mathfrak{X}^*))$ characterizes Dunford-Pettis operators. Clearly, the condition must be some weakening of the strongly regular conditions. Towards this weakening, we now present a form of a L_1-oscillation that is a weakening of the usual L_∞-oscillation.

Definition 2.4. For f in L_1 and A in Σ, the Bocce oscillation of f on A is given by

$$\text{Bocce-osc } f\big|_A \;\equiv\; \frac{\int_A |f - \frac{\int_A f\,d\mu}{\mu(A)}|\,d\mu}{\mu(A)} \quad , \quad \text{where } 0/0 \text{ is } 0\ .$$

Since Bocce-osc $f\big|_A \;\le\;$ osc $f\big|_A$ for any bounded f in L_1 and A in Σ^+, the following definitions are weakenings of the corresponding conditions in Fact 2.2.

Definition 2.5. A subset K of L_1 is a set of small Bocce oscillation if for each $\epsilon > 0$ there is a finite positive measurable partition $\mathcal{P}$ of Ω such that for each f in K

$$\sum_{A\in\mathcal{P}} \mu(A) \text{ Bocce-osc } f\big|_A \;<\; \epsilon\ .$$

Definition 2.6. A subset K of L_1 satisfies the Bocce criterion if for each $\epsilon > 0$ and B in Σ^+ there is a finite collection $\mathcal{F}$ of subset of B each with positive measure such that for each f in K there is an A in $\mathcal{F}$ satisfying

$$\text{Bocce-osc } f\big|_A < \epsilon\ .$$

We now state the result corresponding to Fact 2.2.

Theorem 2.7. *If K is a uniformly bounded subset of L_1, then the following statements are equivalent.*

(1) *K is relatively L_1-compact.*

(2) *K is a set of small Bocce oscillation.*

(3) *K satisfies the Bocce criterion.*

Before passing to the proof of Theorem 2.7, we note that a direct application of this theorem and Fact 1.1 to the uniformly bounded subset $T^*(B(\mathfrak{X}^*))$ of L_1 yields the following characterization of Dunford-Pettis operators, this chapter's main result.

Corollary 2.8. *If T is a bounded linear operator from L_1 into $\mathfrak{X}$, then the following statements are equivalent.*

(1) *T is a Dunford-Pettis operator.*

(2) *$T^*(B(\mathfrak{X}^*))$ is a set of small Bocce oscillation.*

(3) *$T^*(B(\mathfrak{X}^*))$ satisfies the Bocce criterion.*

The remainder of this chapter is devoted to the proof of Theorem 2.7. Because of its length and complexity and also for the sake of the clarity of the exposition, we present the implications in this theorem as three separate theorems.

Theorem 2.10 below shows the equivalence of (1) and (2) in Theorem 2.7.

Remark 2.9. For a finite measurable partition π of Ω, let $E_\pi(f)$ denote the conditional expectation of f relative to the sigma field generated by π. Thus

$$E_\pi(f) = \sum_{B\in\pi} \frac{\int_B f \, d\mu}{\mu(B)} \chi_B \ ,$$

observing the convention that $0/0$ is 0.

Theorem 2.10. *If K is a bounded subset of L_1, then the following statements are equivalent.*

(1) *K is relatively L_1-compact.*

(2) *For each $\epsilon > 0$ there is a finite measurable partition π of Ω such that $\| f - E_\pi(f) \|_{L_1} < \epsilon$ for each f in K.*

(3) *K is a set of small Bocce oscillation.*

Proof. For bounded subsets of L_1, the equivalence of (1) and (2) is well-known and easy to check. The equivalence of (2) and (3) follows directly from

$$\begin{aligned} \| f - E_\pi(f) \|_{L_1} &= \int_\Omega \mid f - \sum_{B\in\pi} \frac{\int_B f \, d\mu}{\mu(B)} \chi_B \mid \, d\mu \\ &= \sum_{B\in\pi} \int_B \mid f - \frac{\int_B f \, d\mu}{\mu(B)} \mid \, d\mu \\ &= \sum_{B\in\pi} \mu(B) \text{ Bocce-osc } f\big|_B \,, \end{aligned}$$

and the definition of a set of small Bocce oscillation. $\square$

Viewed as a function from Σ into the real numbers, Bocce-osc $f\big|_{(\cdot)}$ is not increasing (see Example 2.14); however, $\mu(\cdot)$ Bocce-osc $f\big|_{(\cdot)}$ is increasing (see Remark 2.15). With this in mind, we now proceed with Theorem 2.11 which shows that (2) implies (3) in Theorem 2.7.

Theorem 2.11. *If a subset of L_1 is a set of small Bocce oscillation, then it satisfies the Bocce criterion.*

Proof. Let the subset K of L_1 be a set of small Bocce oscillation. Fix $\epsilon > 0$ and B in Σ^+. Since K is a set of small Bocce oscillation, there is a positive measurable partition $\pi = \{A_1, A_2, \dots, A_n\}$ of Ω such that for each f in K

$$\sum_{i=1}^{n} \mu(A_i) \; (\text{Bocce-osc } f\big|_{A_i}) \quad < \quad \epsilon \; \mu(B) \quad .$$

Set

$$\mathcal{F} = \{A_i \cap B \; : \; A_i \in \pi \quad \text{and} \quad \mu(A_i \cap B) > 0\}.$$

Fix f in K.

Since the function $\mu(\cdot)$ Bocce-osc $f\big|_{(\cdot)}$ is increasing, if Bocce-osc $f\big|_{A_i \cap B} \geq \epsilon$ for each set $A_i \cap B$ in $\mathcal{F}$ then we would have that

$$\epsilon \;\cdot\; \mu(B) \;=\; \sum_{i=1}^{n} \epsilon \;\cdot\; \mu(A_i \cap B)$$

$$
\begin{aligned}
&\le \sum_{i=1}^{n} \mu(A_i \cap B) \text{ Bocce-osc } f\big|_{A_i \cap B} \\
&\le \sum_{i=1}^{n} \mu(A_i) \text{ Bocce-osc } f\big|_{A_i} \\
&< \epsilon \cdot \mu(B) .
\end{aligned}
$$

This cannot be, so there is a set $A_i \cap B$ in $\mathcal{F}$ such that the Bocce-osc $f\big|_{A_i \cap B} < \epsilon$. Thus the set K satisfies the Bocce criterion. □

Theorem 2.12 shows that (3) implies (2) in Theorem 2.7. We shall present the author's original proof of this implication. For a new short proof, we refer the reader to [G3].

Theorem 2.12. *If a uniformly bounded subset of L_1 satisfies the Bocce criterion, then it is relatively L_1-compact.*

We need the following lemma which we will prove after the proof of Theorem 2.12.

Lemma 2.13. *Let the uniformly bounded subset K of L_1 satisfy the Bocce criterion. If $\epsilon > 0$ and $\{f_n\}$ is a sequence in K, then there are disjoint sets $B_1, \ldots, B_p$ in Σ^+ and a subsequence $\{g_n\}$ of $\{f_n\}$ satisfying*

$$
\int_\Omega \left| \, g_n - \sum_{j=1}^{p} \frac{\int_{B_j} g_n \, d\mu}{\mu(B_j)} \chi_{B_j} \, \right| d\mu \;\le 2\,\epsilon \quad . \tag{$**$}
$$

Proof of Theorem 2.12. Let the uniformly bounded subset K of L_1 satisfy the Bocce criterion. Choose a sequence $\{f_n\}$ in K and a sequence $\{\epsilon_k\}$ of positive real numbers decreasing to zero.

It suffices to find a sequence $\{\pi_k\}$ of finite measurable partitions of Ω and a nested sequence $\{\{f_n^k\}_n\}_k$ of subsequences of $\{f_n\}$ such that for all positive integers n and k

$$
\| \, f_n^k - E_k(f_n^k) \, \|_{L_1} \le \; 2 \, \epsilon_k \ , \tag{$*$}
$$

where $E_k(f)$ denotes the conditional expectation of f relative to the σ-field generated by π_k (see Remark 2.9). For then the set $\{f_n^n\}$ is relatively L_1-compact since it can be uniformly approximated in the L_1-norm within $2\epsilon_k$ by the relatively compact set

$$\{E_k(f_n^n)\}_{n\geq k} \cup \{f_n^n\}_{n<k} \ ;$$

hence, there is a L_1-convergent subsequence of $\{f_n\}$.

Towards $(*)$, repeated applications of Lemma 2.13 yield for each positive integer k:

(1) disjoint subsets $B_1^k, \dots, B_{n_k}^k$ of Ω each with positive measure, and

(2) a subsequence $\{f_n^k\}_n$ of $\{f_n^{k-1}\}_n$ (where we write $\{f_n^0\}$ for $\{f_n\}$)

satisfying for each positive integer n

$$\int_\Omega \mid f_n^k - \sum_{j=1}^{n_k} \frac{\int_{B_j^k} f_n^k \, d\mu}{\mu(B_j^k)} \chi_{B_j^k} \mid d\mu \ \leq \ \epsilon_k \ .$$

Set

$$B_0^k = \Omega \setminus \bigcup_{j=1}^{n_k} B_j^k$$

and let π_k be the partition generated by $B_0^k, B_1^k, \dots, B_{n_k}^k$. Still observing the convention that $0/0$ is 0, two appeals to the above inequality yield for each positive integer n and k

$$\begin{aligned}
\| f_n^k - E_k(f_n^k) \|_{L_1} &= \int_\Omega \mid f_n^k - \sum_{j=0}^{n_k} \frac{\int_{B_j^k} f_n^k \, d\mu}{\mu(B_j^k)} \chi_{B_j^k} \mid d\mu \\
&\leq \int_\Omega \mid f_n^k - \sum_{j=1}^{n_k} \frac{\int_{B_j^k} f_n^k \, d\mu}{\mu(B_j^k)} \chi_{B_j^k} \mid d\mu \ + \ \int_\Omega \mid \frac{\int_{B_0^k} f_n^k \, d\mu}{\mu(B_0^k)} \chi_{B_0^k} \mid d\mu \\
&\leq \ \epsilon_k \ + \ \int_{B_0^k} \mid f_n^k \mid d\mu \\
&= \ \epsilon_k \ + \ \int_{B_0^k} \mid f_n^k - \sum_{j=1}^{n_k} \frac{\int_{B_j^k} f_n^k \, d\mu}{\mu(B_j^k)} \chi_{B_j^k} \mid d\mu
\end{aligned}$$

$$\le \ \epsilon_k \ + \int_\Omega \mid f_n^k - \sum_{j=1}^{n_k} \frac{\int_{B_j^k} f_n^k \ d\mu}{\mu(B_j^k)} \chi_{B_j^k} \mid d\mu$$
$$\le \ 2 \ \epsilon_k \ .$$

Thus the proof of Theorem 2.12 is finished as soon as we verify Lemma 2.13.

Proof of Lemma 2.13. Let the uniformly bounded subset K of L_1 satisfy the Bocce criterion. Fix $\epsilon > 0$ and a sequence $\{f_n\}$ in K. The proof is an exhaustion-type argument.

Let $\mathcal{E}_1$ denote the collection of subsets B of Ω such that there are disjoint subsets $B_1, \ldots, B_p$ of B each with positive measure and a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ satisfying

$$\int_B \mid f_{n_k} - \sum_{j=1}^{p} \frac{\int_{B_j} f_{n_k} \ d\mu}{\mu(B_j)} \chi_{B_j} \mid d\mu \ \le \ \epsilon \ \mu(B) \quad .$$

Since K satisfies the Bocce criterion, there is an A in Σ^+ and a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ satisfying

$$\int_A \mid f_{n_k} - \frac{\int_A f_{n_k} \ d\mu}{\mu(A)} \mid d\mu \ \le \ \epsilon \ \mu(A) \quad .$$

Thus the collection $\mathcal{E}_1$ is not empty. If there is a set B in $\mathcal{E}_1$ with corresponding subsets $B_1, \ldots, B_p$ and a subsequence $\{g_n\}$ of $\{f_n\}$ satisfying

$$\int_\Omega \mid g_n - \sum_{j=1}^{p} \frac{\int_{B_j} g_n \ d\mu}{\mu(B_j)} \chi_{B_j} \mid d\mu \ \le \ \epsilon \quad , \tag{1}$$

then we are finished since (1) implies $(**)$.

Otherwise, let j_1 be the smallest positive integer for which there is a C_1 in $\mathcal{E}_1$ with $\frac{1}{j_1} \le \ \mu(C_1)$. Accordingly, there is a finite sequence $\{C_j^1\}$ of disjoint subsets of C_1 each with positive measure and a subsequence $\{f_n^1\}$ of $\{f_n\}$ satisfying

$$\int_{C_1} \mid f_n^1 - \sum_{j} \frac{\int_{C_j^1} f_n^1 \ d\mu}{\mu(C_j^1)} \chi_{C_j^1} \mid d\mu \ \le \ \epsilon \ \mu(C_1) \quad . \tag{2}$$

Note that since condition (1) was not satisfied, $\mu(C_1) < \mu(\Omega) = 1$.

Let $\mathcal{E}_2$ denote the collection of subsets B of $\Omega \setminus C_1$ such that there are disjoint subsets $B_1, \ldots, B_p$ of B each with positive measure and a subsequence $\{f_{n_k}\}$ of $\{f_n^1\}$ satisfying

$$\int_B \mid f_{n_k} - \sum_{j=1}^{p} \frac{\int_{B_j} f_{n_k} \, d\mu}{\mu(B_j)} \chi_{B_j} \mid d\mu \ \leq \ \epsilon \ \mu(B) \quad .$$

Since K satisfies the Bocce criterion and $\mu(\Omega \setminus C_1) > 0$, we see that $\mathcal{E}_2$ is not empty. If there is a set B in $\mathcal{E}_2$ with corresponding finite sequence $\{C_j^2\}$ of subsets of B and a subsequence $\{g_n\}$ of $\{f_n^1\}$ satisfying

$$\int_{\Omega \setminus C_1} \mid g_n - \sum_{j} \frac{\int_{C_j^2} g_n \, d\mu}{\mu(C_j^2)} \chi_{C_j^2} \mid d\mu \ \leq \ \epsilon \ \mu(\Omega \setminus C_1) \quad , \tag{3}$$

then we are finished. For in this case, $(**)$ holds since inequalities (2) and (3) insure that

$$\int_{\Omega} \mid g_n - \sum_{\substack{k=1,2 \\ j}} \frac{\int_{C_j^k} g_n \, d\mu}{\mu(C_j^k)} \chi_{C_j^k} \mid d\mu \leq \ \epsilon \ \mu(C_1) + \epsilon \ \mu(\Omega \setminus C_1) \ = \ \epsilon \ .$$

Otherwise, let j_2 be the smallest positive integer for which there is a C_2 in $\mathcal{E}_2$ with $\frac{1}{j_2} \leq \ \mu(C_2)$. Accordingly, there is a finite sequence $\{C_j^2\}$ of disjoint subsets of C_2 each with positive measure and a subsequence $\{f_n^2\}$ of $\{f_n^1\}$ satisfying

$$\int_{C_2} \mid f_n^2 - \sum_{j} \frac{\int_{C_j^2} f_n^2 \, d\mu}{\mu(C_j^2)} \chi_{C_j^2} \mid d\mu \ \leq \ \epsilon \ \mu(C_2) \quad .$$

Note that since condition (3) was not satisfied, $\mu(C_2) < \mu(\Omega \setminus C_1)$.

Continue in this way. If the process stops in a finite number of steps then we are finished. If the process does not stop, then diagonalize the resultant sequence $\{ \ \{f_n^k\}_{n=1}^{\infty} \ \}_{k=1}^{\infty}$ of sequences to obtain the sequence $\{f_n^n\}_{n=1}^{\infty}$ and set $C_\infty = \ \bigcup_{k=1}^{\infty} C_k$.

Note that $\mu(\Omega \setminus C_\infty) = 0$. For if $\mu(\Omega \setminus C_\infty) > 0$ then, since K satisfies the Bocce criterion, there is a subset B of $\Omega \setminus C_\infty$ with positive measure and a subsequence $\{h_n\}$ of $\{f_n^n\}$ satisfying

$$\int_B \mid h_n - \frac{\int_B h_n \, d\mu}{\mu(B)} \mid d\mu \;<\; \epsilon \, \mu(B) \; .$$

Since for each positive integer m

$$\sum_{k=1}^{m} \frac{1}{j_k} \;\leq\; \mu(\bigcup_{k=1}^{m} C_k) \;\leq\; \mu(\Omega)$$

and $j_m > 1$, we can choose an integer $m > 1$ such that

$$\frac{1}{j_m - 1} \leq \mu(B) \; .$$

But this implies that B is in $\mathcal{E}_m$ since

$$B \subset \Omega \setminus C_\infty \subset \Omega \setminus \bigcup_{k=1}^{m-1} C_k$$

and $\{h_n\}_{n=m-1}^{\infty}$ is a subsequence of $\{f_n^n\}_{n=m-1}^{\infty}$, which in turn is a subsequence of $\{f_n^{m-1}\}_{n=1}^{\infty}$. This contradicts the choice of j_m. Thus $\mu(\Omega \setminus C_\infty) = 0$.

Since K is uniformly bounded, there is a real number M such that $\| \; f \; \|_\infty \leq M$ for each f in K. Pick an integer m so that

$$M \;\mu(\Omega \setminus \bigcup_{k=1}^{m} C_k) \;\leq\; \epsilon \; .$$

Since $\{f_n^n\}_{n=m}^{\infty}$ is a subsequence of $\{f_n^m\}_{n=1}^{\infty}$, for each f in $\{f_n^n\}_{n=m}^{\infty}$ we have

$$\begin{aligned}
&\int_\Omega \mid f - \sum_{\substack{1 \leq k \leq m \\ j}} \frac{\int_{C_j^k} f \; d\mu}{\mu(C_j^k)} \chi_{C_j^k} \mid d\mu \\
&\quad = \sum_{k=1}^{m} \int_{C_k} \mid f - \sum_j \frac{\int_{C_j^k} f \; d\mu}{\mu(C_j^k)} \chi_{C_j^k} \mid d\mu \;+\; \int_{\Omega \setminus \bigcup_{k=1}^{m} C_k} |f| \; d\mu \\
&\quad \leq \sum_{k=1}^{m} \epsilon \; \mu(C_k) \;+\; M\mu(\Omega \setminus \bigcup_{k=1}^{m} C_k) \\
&\quad \leq \epsilon \;+\; \epsilon \;\;=\;\; 2\,\epsilon \; .
\end{aligned}$$

So the disjoint subsets $\{C_j^k : j \geq 1 \text{ and } k = 1, \dots, m\}$ along with the subsequence $\{f_n^n\}_{n=m}^{\infty}$ of $\{f_n\}$ satisfy the conditions of the lemma. $\square$

This completes the proof of Theorem 2.7.

The proof that if K has the Bourgain property then K is a set of small oscillation uses the fact that the function osc $f\big|_{(\cdot)}$ from Σ into the real numbers is increasing [cf. GGMS]. The proof of Theorem 2.12 is complicated by the fact that the function Bocce-osc $f\big|_{(\cdot)}$ does not enjoy this property, as our next example illustrates.

Example 2.14. Let $A = [0, 1/4]$ and $B = [0, 1]$. Define the function f from [0,1] into the real numbers by $f(t) = \chi_C(t)$ where $C = [1/8, 1]$. It is straightforward to verify that Bocce-osc $f\big|_A = 1/2$ but Bocce-osc $f\big|_B = 7/32$.

Remark 2.15. In the proof of Theorem 2.11, we used the fact that the function

$$\mu(\cdot) \text{ Bocce-osc } f\big|_{(\cdot)} \;\equiv\; \int_{(\cdot)} \left| \, f - \frac{\int_{(\cdot)} f \, d\mu}{\mu(\cdot)} \, \right| \, d\mu$$

is an increasing function from Σ into the reals numbers. To see this, let A be a subset of B with A and B in Σ^+. Let

$$m_A = \frac{\int_A f \, d\mu}{\mu(A)} \quad \text{and} \quad m_B = \frac{\int_B f \, d\mu}{\mu(B)} \; .$$

Note that

$$\begin{aligned} &\int_A |\, f - m_A \,| \; d\mu \\ &\qquad \leq \int_A |\, f - m_B \,| \; d\mu \;+\; \int_A |\, m_B - m_A \,| \; d\mu \\ &\qquad = \int_B |\, f - m_B \,| \; d\mu \;-\; \int_{B \setminus A} |\, f - m_B \,| \; d\mu \;+\; \int_A |\, m_B - m_A \,| \; d\mu \; . \end{aligned}$$

Thus

$$\int_A |\, f - m_A \,| \; d\mu \;\leq\; \int_B |\, f - m_B \,| \; d\mu$$

since

$$\begin{aligned}
\int_A \mid m_B - m_A \mid \, d\mu &= \mu(A) \mid m_B - m_A \mid \\
&= \mid \frac{\mu(A)}{\mu(B)} \int_B f \, d\mu \; - \; \int_A f \, d\mu \mid \\
&= \mid \frac{\mu(A)}{\mu(B)} \int_B f \, d\mu \; - \Big(\int_B f \, d\mu \; - \; \int_{B \setminus A} f \, d\mu \Big) \mid \\
&= \mid \int_{B \setminus A} f \, d\mu \; - \; \frac{\mu(B) - \mu(A)}{\mu(B)} \int_B f \, d\mu \mid \\
&= \mid \int_{B \setminus A} (f - m_B) \, d\mu \mid \; .
\end{aligned}$$

Remark 2.16. Fix $f \in L_1$. Viewed as a function from Σ into the real numbers, Bocce-osc $f\big|_{(\cdot)}$ is continuous, i.e. if a sequence $\{A_n\}$ in Σ converges to $A \in \Sigma$ in the sense that $\mu(A \triangle A_n)$ tends to 0, then Bocce-osc $f\big|_{A_n}$ tends to Bocce-osc $f\big|_{A}$. We can verify this using methods similar to the methods in Remark 2.15; or we can note that this is an easy consequence of the Lebesgue Dominated Convergence Theorem.

Remark 2.17. If a subset K of L_1 satisfies the Bocce criterion, then the translate of K by a L_1-function f also satisfies the Bocce criterion.

If K is uniformly bounded and f is bounded, then this is an easy consequence of Theorem 2.7. For the general case, let the subset K of L_1 satisfy the Bocce criterion and $f \in L_1$. We shall to show that the set $K + f \equiv \{ \; g + f \; : \; g \in K \; \}$ satisfies the Bocce criterion. Towards this end, fix $\epsilon > 0$ and $B \in \Sigma^+$. Find $B_0 \subset B$ with $B_0 \in \Sigma^+$ such that f is bounded on B_0.

Approximate $f\chi_{B_0}$ in L_∞-norm within $\frac{\epsilon}{4}$ by a simple function $\tilde{f}$. Find $C \subset B_0$ with $C \in \Sigma^+$ such that $\tilde{f}$ is constant on C. Since K satisfies the Bocce criterion, we can find a finite collection $\mathcal{F}$ of subsets corresponding to $\frac{\epsilon}{2}$ and C.

Fix $g + f \in K + f$. Find $A \in \mathcal{F}$ such that Bocce-osc $g\big|_A < \frac{\epsilon}{2}$. Note that since

$\tilde{f}$ is constant on A, Bocce-osc $g\big|_A = $ Bocce-osc $(g+\tilde{f})\big|_A$. Compute,

$$\begin{aligned} \text{Bocce-osc } (g+f)\big|_A &\le \text{Bocce-osc } (g+\tilde{f})\big|_A + \text{Bocce-osc } (\tilde{f}-f)\big|_A \\ &\le \text{Bocce-osc } g\big|_A + 2\ \|\,(\tilde{f}-f)\,\chi_A\,\|_{L_\infty} \\ &< \epsilon\ . \end{aligned}$$

Thus $K + f$ satisfies the Bocce criterion.

CHAPTER 3

DENTABILITY

In this chapter, we examine in which Banach spaces bounded subsets have certain dentability properties.

Dentability characterizations of the RNP are well-known [cf. DU and GU].

Fact 3.1. The following statements are equivalent.

(1) $\mathfrak{X}$ has the RNP.

(2) Every bounded subset D of $\mathfrak{X}$ is dentable.

Definition 3.2 D is <u>dentable</u> if for each $\epsilon > 0$ there is x in D such that $x \notin \overline{\text{co}}\ (D \setminus B_\epsilon(x))$ where $B_\epsilon(x) = \{y \in \mathfrak{X} : \| x - y \| < \epsilon\}$.

(3) Every bounded subset D of $\mathfrak{X}$ is σ-dentable.

Definition 3.3 D is <u>σ-dentable</u> if for each $\epsilon > 0$ there is an x in D such that if x has the form $x = \sum_{i=1}^n \alpha_i z_i$ with $z_i \in D,\ \ 0 \le \alpha_i$, and $\sum_{i=1}^n \alpha_i = 1$, then $\| x - z_i \| < \epsilon$ for some i .

(4) Every bounded subset D of $\mathfrak{X}$ has slices of arbitrarily small diameter.

A weakening of condition (4) provides a dentability characterization of strongly regular spaces [cf. GGMS]. This characterization is often used as the definition.

Fact 3.4. $\mathfrak{X}$ is strongly regular if and only if for each bounded subset D of $\mathfrak{X}$ and $\epsilon > 0$ there are finitely many slices $S_1, \dots , S_n$ of D such that

$$\text{diam} \left\{ \frac{S_1 + \ldots + S_n}{n} \right\} < \epsilon \ .$$

The natural question to explore next is what dentability condition characterizes the CCP. Towards this, the next definition is a weakening of Definition 3.2.

Definition 3.5. A subset D of $\mathfrak{X}$ is weak-norm-one dentable if for each $\epsilon > 0$ there is a finite subset F of D such that for each x^* in $S(\mathfrak{X}^*)$ there is x in F satisfying

$$x \notin \overline{\text{co}}\ \{z \in D : \mid x^*(z-x) \mid \geq \epsilon\} \ \equiv\ \overline{\text{co}}\ (D \setminus V_{\epsilon, x^*}(x))\ .$$

Petrakis and Uhl [PU] showed that if $\mathfrak{X}$ has the CCP then every bounded subset of $\mathfrak{X}$ is weak-norm-one dentable. For our characterization of the CCP, we introduce the following variations of definition 3.3 that are useful in showing the converse of the above implication of [PU].

Definition 3.6. A subset D of $\mathfrak{X}$ is Bocce dentable if for each $\epsilon > 0$ there is a finite subset F of D such that for each x^* in $S(\mathfrak{X}^*)$ there is x in F satisfying:

$$\text{if}\ \ x = \sum_{i=1}^{n} \alpha_i z_i \quad \text{with } z_i \in D, \ \ 0 \leq \alpha_i, \ \ \text{and } \sum_{i=1}^{n} \alpha_i = 1\ ,$$

$$\text{then} \ \sum_{i=1}^{n} \ \alpha_i \mid x^*(x - z_i) \mid < \epsilon.$$

Definition 3.7. A subset D of $\mathfrak{X}$ is midpoint Bocce dentable if for each $\epsilon > 0$ there is a finite subset F of D such that for each x^* in $S(\mathfrak{X}^*)$ there is x in F satisfying:

$$\text{if} \quad x = \tfrac{1}{2} z_1 + \tfrac{1}{2} z_2 \quad \text{with } z_i \in D$$

$$\text{then} \quad \mid x^*(x - z_1) \mid \equiv \mid x^*(x - z_2) \mid < \epsilon.$$

We obtain equivalent formulations of the above definitions by replacing $S(\mathfrak{X}^*)$ with $B(\mathfrak{X}^*)$. The next theorem, this chapter's main result, shows that these dentability conditions provide an internal geometric characterization of the CCP.

Theorem 3.8. *The following statements are equivalent.*

(1) $\mathfrak{X}$ *has the CCP.*

(2) *Every bounded subset of* $\mathfrak{X}$ *is weak-norm-one dentable.*

(3) *Every bounded subset of* $\mathfrak{X}$ *is midpoint Bocce dentable.*

(4) *Every bounded subset of* $\mathfrak{X}$ *is Bocce dentable.*

The remainder of this chapter is devoted to the proof of Theorem 3.8. Because of its length and complexity and also for the sake of clarity of the exposition, we present the implications as separate theorems. It is clear from the definitions that (2) implies (3) and that (4) implies (3). For completeness sake, Fact 3.14 presents the proof of [PU] that (1) implies (2). Using Corollary 2.8, Theorem 3.11 shows that (3) implies (1). That (1) implies (4) follows from Theorem 3.9 and the martingale characterization (Fact 1.3) of the CCP.

Theorem 3.9. *If a subset* D *of* $\mathfrak{X}$ *is not Bocce dentable, then there is an increasing sequence* $\{\pi_n\}$ *of partition of* $[0,1)$ *and a* D*-valued martingale* $\{f_n, \sigma(\pi_n)\}$ *that is not Cauchy in the Pettis norm. Moreover,* $\{\pi_n\}$ *can be chosen so that* $\vee\, \sigma(\pi_n) = \Sigma$*,* $\pi_0 = \{\Omega\}$ *and each* π_n *partitions* $[0,1)$ *into a finite number of half-open intervals.*

Proof. Let D be a non-Bocce-dentable subset of $\mathfrak{X}$. Accordingly, there is $\epsilon > 0$ satisfying:

$$\begin{gathered}\text{for each finite subset } F \text{ of } D \text{ there is } x_F^* \text{ in } S(\mathfrak{X}^*) \text{ such that}\\ \text{each } x \text{ in } F \text{ has the form} \quad x = \sum_{i=1}^{m} \alpha_i z_i \\ \text{with} \quad \sum_{i=1}^{m} \alpha_i \mid x_F^*(x - z_i) \mid > \epsilon \\ \text{for a suitable choice of} \quad z_i \in D \quad \text{and} \quad \alpha_i > 0 \quad \text{with} \quad \sum_{i=1}^{m} \alpha_i = 1 \ .\end{gathered} \tag{$*$}$$

We shall use property $(*)$ to construct an increasing sequence $\{\pi_n\}_{n\geq 0}$ of finite partitions of $[0,1)$, a martingale $\{f_n, \sigma(\pi_n)\}_{n\geq 0}$, and a sequence $\{x_n^*\}_{n\geq 1}$ in $S(\mathfrak{X}^*)$ such that for each nonnegative integer n:

(1) f_n has the form $f_n = \sum_{E\in\pi_n} x_E \chi_E$ where x_E is in D,

(2) $\int_\Omega \mid x_{n+1}^*(f_{n+1} - f_n) \mid d\mu \geq \epsilon$,

(3) if E is in π_n, then E has the form $[a,b)$ and $\mu(E) < 1/2^n$ and

(4) $\pi_0 = \{\Omega\}$.

Condition (3) guarantees that $\vee\ \sigma(\pi_n) = \Sigma$ while condition (2) guarantees that $\{f_n\}$ is not Cauchy in the Pettis norm.

Towards the construction, pick an arbitrary x in D. Set $\pi_0 = \{\Omega\}$ and $f_0 = x\,\chi_\Omega$. Fix $n \geq 0$. Suppose that a partition π_n of Ω consisting of intervals of length at most $1/2^n$ and a function $f_n = \sum_{E\in\pi_n} x_E\chi_E$ with $x_E \in D$ have been constructed. We now construct f_{n+1}, π_{n+1} and x_{n+1}^* satisfying conditions (1), (2) and (3).

Apply $(*)$ to $F = \{x_E\, :\, E \in \pi_n\}$ and find the associated $x_F^* = x_{n+1}^*$ in $S(\mathfrak{X}^*)$. Fix an element $E = [a,b)$ of π_n. We first define $f_{n+1}\chi_E$. Property $(*)$ gives that x_E has the form

$$x_E = \sum_{i=1}^{m} \alpha_i x_i$$

with

$$\sum_{i=1}^{m} \alpha_i \mid x_{n+1}^*(x - x_i) \mid > \ \epsilon$$

for a suitable choice of $x_i \in D$ and positive real numbers $\alpha_1, \dots, \alpha_m$ whose sum is one. Using repetition, we arrange to have $\alpha_i < 1/2^{n+1}$ for each i. It follows that there are real numbers $d_0, d_1, \dots, d_m$ such that

$$a = d_0 < d_1 < \ldots < d_{m-1} < d_m = b$$

and

$$d_i - d_{i-1} = \alpha_i\ (b-a) \quad \text{for} \quad i = 1, \dots, m \quad .$$

Set

$$f_{n+1}\chi_E \;=\; \sum_{i=1}^{m} x_i \; \chi_{[d_{i-1},d_i)} \; .$$

Define f_{n+1} on all of Ω similarly. Let π_{n+1} be the partition consisting of all the intervals $[d_{i-1}, d_i)$ obtained from letting E range over π_n.

Clearly, f_{n+1} and π_{n+1} satisfy conditions (1) and (3). Condition (2) is also satisfied since for each $E = [a, b)$ in π_n we have, using the above notation,

$$\begin{aligned}\int_E \mid x_{n+1}^*(f_{n+1} - f_n) \mid \, d\mu &= \sum_{i=1}^{m} \int_{d_{i-1}}^{d_i} \mid x_{n+1}^*(x_i - x_E) \mid \, d\mu \\ &= (b-a) \sum_{i=1}^{m} \alpha_i \mid x_{n+1}^*(x_i - x_E) \mid \\ &\geq \mu(E)\,\epsilon \; .\end{aligned}$$

To insure that $\{f_n\}$ is indeed a martingale, we need to compute $E_n(f_{n+1})$. Fix $E = [a, b)$ in π_n. Using the above notation, we have for almost all t in E,

$$\begin{aligned}E_n(f_{n+1})(t) \;&=\; \frac{1}{b-a} \int_a^b f_{n+1}\, d\mu \\ &=\; \frac{1}{b-a} \sum_{i=1}^{m} \int_{d_{i-1}}^{d_i} f_{n+1}\, d\mu \\ &=\; \sum_{i=1}^{m} \frac{d_i - d_{i-1}}{b-a}\, x_i \\ &=\; \sum_{i=1}^{m} \alpha_i\, x_i \;=\; x_E \\ &=\; f_n(t) \; .\end{aligned}$$

Thus $E_n(f_{n+1}) = f_n$ a.e., as needed.

This completes the necessary constructions. $\square$

We need the following lemma which we will prove after the proof of Theorem 3.11.

Lemma 3.10. *If A is in Σ^+ and f in $L_\infty(\mu)$ is not constant a.e. on A, then there is an increasing sequence $\{\pi_n\}$ of positive finite measurable partitions of A such that*

$\vee\ \sigma(\pi_n) = \Sigma \cap A$ *and for each* n

$$\mu\left(\bigcup\left\{E \quad : \quad E \in \pi_n, \quad \frac{\int_E f\,d\mu}{\mu(E)} \geq \frac{\int_A f\,d\mu}{\mu(A)}\ \right\}\right) = \frac{\mu(A)}{2}\ ,$$

and so

$$\mu\left(\bigcup\left\{E \quad : \quad E \in \pi_n, \quad \frac{\int_E f\,d\mu}{\mu(E)} < \frac{\int_A f\,d\mu}{\mu(A)}\ \right\}\right) = \frac{\mu(A)}{2}\ .$$

Theorem 3.11. *If all bounded subsets of* $\mathfrak{X}$ *are midpoint Bocce dentable, then* $\mathfrak{X}$ *has the complete continuity property.*

Proof. Let all bounded subsets of $\mathfrak{X}$ be midpoint Bocce dentable. Fix a bounded linear operator T from L_1 into $\mathfrak{X}$. We shall show that the subset $T^*(B(\mathfrak{X}^*))$ of L_1 satisfies the Bocce criterion. Then an appeal to Corollary 2.8 shows that $\mathfrak{X}$ has the complete continuity property.

To this end, fix $\epsilon > 0$ and B in Σ^+. Let F denote the vector measure from Σ into $\mathfrak{X}$ given by $F(E) = T(\chi_E)$. Since the subset $\{\frac{F(E)}{\mu(E)} : E \subset B$ and $E \in \Sigma^+\}$ of $\mathfrak{X}$ is bounded, it is midpoint Bocce dentable. Accordingly, there is a finite collection $\mathcal{F}$ of subsets of B each in Σ^+ such that for each x^* in the unit ball of $\mathfrak{X}^*$ there is a set A in $\mathcal{F}$ such that if

$$\frac{F(A)}{\mu(A)} \quad = \quad \frac{1}{2}\ \frac{F(E_1)}{\mu(E_1)} \quad + \quad \frac{1}{2}\ \frac{F(E_2)}{\mu(E_2)}$$

for some subsets E_i of B with $E_i \in \Sigma^+$, then

$$\frac{1}{2}\left|\frac{x^*F(E_1)}{\mu(E_1)} - \frac{x^*F(A)}{\mu(A)}\right| \quad + \quad \frac{1}{2}\left|\frac{x^*F(E_2)}{\mu(E_2)} - \frac{x^*F(A)}{\mu(A)}\right| \quad < \quad \epsilon\ . \tag{1}$$

Fix x^* in the unit ball of $\mathfrak{X}^*$ and find the associated A in $\mathcal{F}$. By definition, the set $T^*(B(\mathfrak{X}^*))$ will satisfy the Bocce criterion provided that Bocce-osc $(T^*x^*)\big|_A \leq \epsilon$.

If the L_1-function T^*x^* is constant a.e. on A, then the Bocce-osc $(T^*x^*)\big|_A$ is zero and we are finished. So assume T^*x^* is not constant a.e. on A.

For a finite positive measurable partition π of A, denote

$$f_\pi = \sum_{E\in\pi} \frac{F(E)}{\mu(E)}\chi_E$$

and

$$E_\pi^+ = \bigcup\left\{ E\in\pi \,:\, \frac{x^*F(E)}{\mu(E)} \geq \frac{x^*F(A)}{\mu(A)} \right\}$$

and

$$E_\pi^- = \bigcup\left\{ E\in\pi \,:\, \frac{x^*F(E)}{\mu(E)} < \frac{x^*F(A)}{\mu(A)} \right\} .$$

Note that for E in Σ

$$x^*F(E) = \int_E (x^*T^*)\, d\mu \ .$$

Compute

$$\begin{aligned}
&\int_A \left| x^* f_\pi - \frac{x^*F(A)}{\mu(A)} \right| d\mu \\
&\quad = \sum_{E\in\pi} \int_E \left| \frac{x^*F(E)}{\mu(E)} - \frac{x^*F(A)}{\mu(A)} \right| d\mu \\
&\quad = \mu(A) \sum_{E\in\pi} \frac{\mu(E)}{\mu(A)} \left| \frac{x^*F(E)}{\mu(E)} - \frac{x^*F(A)}{\mu(A)} \right| \qquad (2) \\
&\quad = \mu(A) \left[\frac{\mu(E_\pi^+)}{\mu(A)} \left| \frac{x^*F(E_\pi^+)}{\mu(E_\pi^+)} - \frac{x^*F(A)}{\mu(A)} \right| + \frac{\mu(E_\pi^-)}{\mu(A)} \left| \frac{x^*F(E_\pi^-)}{\mu(E_\pi^-)} - \frac{x^*F(A)}{\mu(A)} \right| \right] .
\end{aligned}$$

Since the L_1-function T^*x^* is bounded, for now we may view T^*x^* as an element in L_∞. Lemma 3.10 allow us to apply property (1) to equation (2). For applying Lemma 3.10 to A with $f \equiv T^*x^*$ produces an increasing sequence $\{\pi_n\}$ of positive measurable partitions of A satisfying

$$\vee\ \sigma(\pi_n) = \Sigma\cap A \qquad \text{and} \qquad \mu(E_{\pi_n}^+) = \frac{\mu(A)}{2} = \mu(E_{\pi_n}^-) \quad .$$

For $\pi = \pi_n$, condition (2) becomes

$$\begin{aligned}
&\int_A \left| x^* f_{\pi_n} - \frac{x^* F(A)}{\mu(A)} \right| d\mu \\
&= \mu(A) \left[\frac{1}{2} \left| \frac{x^* F(E^+_{\pi_n})}{\mu(E^+_{\pi_n})} - \frac{x^* F(A)}{\mu(A)} \right| + \frac{1}{2} \left| \frac{x^* F(E^-_{\pi_n})}{\mu(E^-_{\pi_n})} - \frac{x^* F(A)}{\mu(A)} \right| \right] . \qquad (3)
\end{aligned}$$

Since $F(A) \,/\, \mu(A)$ has the form

$$\frac{F(A)}{\mu(A)} = \frac{\mu(E^+_{\pi_n})}{\mu(A)} \frac{F(E^+_{\pi_n})}{\mu(E^+_{\pi_n})} + \frac{\mu(E^-_{\pi_n})}{\mu(A)} \frac{F(E^-_{\pi_n})}{\mu(E^-_{\pi_n})} = \frac{1}{2} \frac{F(E^+_{\pi_n})}{\mu(E^+_{\pi_n})} + \frac{1}{2} \frac{F(E^-_{\pi_n})}{\mu(E^-_{\pi_n})} ,$$

applying property (1) to equation (3) yields that for each π_n

$$\int_A \left| x^* f_{\pi_n} - \frac{x^* F(A)}{\mu(A)} \right| d\mu \; < \; \mu(A)\, \epsilon \quad .$$

Since $\vee\, \sigma(\pi_n) = \Sigma \cap A$ and

$$(x^* f_{\pi_n})\big|_A = \sum_{E \in \pi_n} \frac{x^* F(E)}{\mu(E)} \chi_E = \sum_{E \in \pi_n} \frac{\int_E (T^* x^*)\, d\mu}{\mu(E)} \chi_E = E_{\pi_n}(T^* x^*)\big|_A ,$$

we have that $(x^* f_{\pi_n})\big|_A$ converges to $(T^* x^*)\big|_A$ in L_1-norm . Hence,

$$\text{Bocce-osc } (T^* x^*)\big|_A \equiv \frac{\int_A \left| (T^* x^*) - \frac{\int_A (T^* x^*)\, d\mu}{\mu(A)} \right| d\mu}{\mu(A)} \; \leq \; \epsilon.$$

Thus $T^*(B(\mathfrak{X}^*))$ satisfies the Bocce criterion and so, as needed, $\mathfrak{X}$ has the complete continuity property. $\square$

We now verify Lemma 3.10.

Proof of Lemma 3.10. Fix A in Σ^+ and f in $L_\infty(\mu)$. Without loss of generality, f is not constant a.e. on A and $\int_A f\, d\mu = 0$. Find P and N in Σ satisfying

$$A = P \cup N \qquad \mu(P) = \frac{\mu(A)}{2} = \mu(N) \qquad P \cap N = \emptyset$$

and

$$\int_P f\, d\mu \; \equiv \; 2M > 0 \qquad \int_N f\, d\mu \; \equiv \; -2M < 0 \quad .$$

Approximate f by a simple function $\tilde{f}(\cdot) = \sum \alpha_i \, \chi_{A_i}(\cdot)$ satisfying

(1) $\| f - \tilde{f} \|_{L_\infty} < M$,

(2) $\cup A_i = A$ and the A_i are disjoint,

(3) $A_i \subset P$ if $i \leq m$ and $A_i \subset N$ if $i > m$ for some positive integer m.

Note that

$$P = \bigcup_{i \leq m} A_i \qquad \text{and} \qquad N = \bigcup_{i > m} A_i \quad .$$

To find the sequence $\{\pi_n\}$, we shall first find an increasing sequence $\{\pi_n^P\}$ of partitions of P and an increasing sequence $\{\pi_n^N\}$ of partitions of N. Then π_n will be the union of π_n^P and π_n^N . To this end, for each A_i obtain an increasing sequence of partitions of A_i :

$$\begin{array}{ccccccc}
 & & & A_i \equiv E_1^{i0} & & & \\
 & & \swarrow & & \searrow & & \\
 & E_1^{i1} & & & & E_2^{i1} & \\
 \swarrow & & \searrow & & \swarrow & & \searrow \\
 E_1^{i2} & & E_2^{i2} & & E_3^{i2} & & E_4^{i2} \\
 & & & \cdots & & &
\end{array}$$

such that for $n = 0, 1, 2, \ldots$ and $k = 1, \ldots , 2^n$

$$E_{2k-1}^{i\;n+1} \cup E_{2k}^{i\;n+1} = E_k^{in} \qquad E_{2k-1}^{i\;n+1} \cap E_{2k}^{i\;n+1} = \emptyset \qquad \mu(E_k^{in}) = \frac{\mu(A_i)}{2^n} \quad .$$

For each positive integer n , let π_n^P be the partition of P given by

$$\pi_n^P = \{P_k^n \, : \, k = 1, \ldots , 2^n\} \qquad \text{where} \qquad P_k^n = \bigcup_{i \leq m} E_k^{in} \ ,$$

π_n^N be the partition of N given by

$$\pi_n^N = \{N_k^n \, : \, k = 1, \ldots , 2^n\} \qquad \text{where} \qquad N_k^n = \bigcup_{i > m} E_k^{in} \ ,$$

and π_n be the partition of A given by

$$\pi_n = \quad \pi_n^P \ \cup \ \pi_n^N \ .$$

The sequence $\{\pi_n\}$ has the desired properties. Since

$$\mu(P_k^n) \ = \ \sum_{i \le m} \frac{\mu(A_i)}{2^n} \ = \ \frac{\mu(P)}{2^n} \ = \ \frac{\mu(A)}{2^{n+1}}$$

and

$$\mu(N_k^n) \ = \ \sum_{i < m} \frac{\mu(A_i)}{2^n} \ = \ \frac{\mu(N)}{2^n} \ = \ \frac{\mu(A)}{2^{n+1}} \ ,$$

any element in π_n has measure $\mu(A) \ / \ 2^{n+1}$. Thus $\vee \ \sigma(\pi_n) = \Sigma \cap A$.

As for the other properties, since $\tilde{f}$ takes the value α_i on $E_k^{in} \subset A_i$ we have

$$\begin{aligned}
\int_{P_k^n} \tilde{f} \, d\mu \ &= \ \sum_{i \le m} \int_{E_k^{in}} \tilde{f} \, d\mu \\
&= \ \sum_{i \le m} \ \alpha_i \ \mu(E_k^{in}) \\
&= \ \frac{1}{2^n} \sum_{i \le m} \ \alpha_i \ \mu(A_i) \\
&= \ \frac{1}{2^n} \int_P \tilde{f} \, d\mu \\
&> \ 0
\end{aligned}$$

and likewise

$$\int_{N_k^n} \tilde{f} \, d\mu \ = \ \frac{1}{2^n} \int_N \tilde{f} \, d\mu \ < \ 0 \ .$$

We chose $\tilde{f}$ close enough to f so that the above inequalities still hold when we replace $\tilde{f}$ by f,

$$\begin{aligned}
\int_{P_k^n} f\,d\mu &\geq \int_{P_k^n} (\tilde{f} - M)\,d\mu \\
&= \frac{1}{2^n}\int_P \tilde{f}\,d\mu \quad - \quad M\mu(P_k^n) \\
&\geq \frac{1}{2^n}\int_P (f - M)\,d\mu \quad - \quad \frac{M\mu(A)}{2^{n+1}} \\
&= \frac{1}{2^n}\int_P f\,d\mu \quad - \quad \frac{M\mu(A)}{2^{n+1}} \quad - \quad \frac{M\mu(A)}{2^{n+1}} \\
&> \frac{M}{2^n} \quad - \quad \frac{M\mu(A)}{2^n} \quad = \quad \frac{M\,[1-\mu(A)]}{2^n} \\
&\geq 0
\end{aligned}$$

and likewise

$$\int_{N_k^n} f\,d\mu \;<\; \frac{M\,[\mu(A) - 1]}{2^n} \;\leq\; 0\,.$$

Thus the other properties of the lemma are satisfied since for each n,

$$\begin{aligned}
\mu\left(\bigcup\left\{E \quad : \quad E \in \pi_n, \quad \int_E f\,d\mu \geq 0\ \right\}\right) &= \quad \mu\left(\bigcup\left\{E \quad : \quad E \in \pi_n^P\ \right\}\right) \\
&= \quad \mu(P) \\
&= \quad \frac{\mu(A)}{2}
\end{aligned}$$

and so

$$\mu\left(\bigcup\left\{E \quad : \quad E \in \pi_n, \quad \int_E f\,d\mu < 0\ \right\}\right) = \frac{\mu(A)}{2}\,.$$

Note that the partitions $\{\pi_n\}$ are nested by construction. $\square$

[PU, Theorem II.7] shows that (1) implies (2) in Theorem 3.8 by constructing, in a bounded non-weak-norm-one dentable subset D, a $(\overline{\text{co}}\ D)$-valued martingale that is not Cauchy in the Pettis norm. Since quasi-martingales enter into their proof, let us pause to recall a few definitions and facts. Fix the notation as in Definition 1.2. Let $\{\epsilon_n\}_{n\geq 0}$ be a summable sequence of non-negative numbers.

Definition 3.12. A sequence $\{f_n\}_{n\geq 0}$ in $L_1(\mathfrak{X})$ is a $\mathfrak{X}$-valued quasi-martingale with respect to $\{\mathcal{F}_n\}$ and corresponding to $\{\epsilon_n\}$ if for each n we have that f_n is $\mathcal{F}_n$-measurable and $\| \, E_n(f_{n+1}) \; - \; f_n \, \|_{L_1} \, \leq \, \epsilon_n$.

A self-contained presentation of quasi-martingales along with the decomposition theorem below may be found in [KR].

Fact 3.13 quasi-martingale decomposition theorem. Let K be a closed bounded convex subset of $\mathfrak{X}$. If $\{f_n, \mathcal{F}_n\}$ is a K-valued quasi-martingale corresponding to $\{\epsilon_n\}$, then there is a K-valued martingale $\{g_n, \mathcal{F}_n\}$ satisfying

$$\| \, f_n - g_n \, \|_{L_1} \, \leq \, \sum_{j=n}^{\infty} \epsilon_j \; .$$

Note that a quasi-martingale is Pettis-Cauchy if and only if the corresponding martingale from the decomposition theorem is Pettis-Cauchy.

Fact 3.14 [PU, Theorem II.7]. Let $\{\epsilon_n\}_{n\geq 0}$ be a summable sequence of positive numbers. If a subset D of $\mathfrak{X}$ is not weak-norm-one dentable, then there is an increasing sequence $\{\pi_n\}$ of partition of $[0,1)$ and a D-valued quasi-martingale $\{f_n, \sigma(\pi_n)\}$ corresponding to $\{\epsilon_n\}$ that is not Cauchy in the Pettis norm. Moreover, $\{\pi_n\}$ can be chosen so that $\vee\, \sigma(\pi_n) = \Sigma$, $\pi_0 = \{\Omega\}$ and each π_n partitions $[0,1)$ into a finite number of half-open intervals. Consequently, if there is a bounded non-weak-norm-one dentable subset D of $\mathfrak{X}$, then there is a $(\overline{\text{co}}\ D)$-valued martingale that is not Pettis-Cauchy.

Proof. Fix a summable sequence $\{\epsilon_n\}_{n\geq 0}$ of positive numbers. Let D be subset of

$\mathfrak{X}$ that is not weak-norm-one dentable. Accordingly, there is an $\epsilon > 0$ satisfying:

$$\begin{array}{c}\text{for each finite subset } F \text{ of } D \text{ there is an } \; x_F^* \text{ in } S(\mathfrak{X}^*) \text{ such that} \\ \text{if } x \text{ is in } F \;\; \text{then} \;\; x \in \overline{\mathrm{co}}\left(z \in D \,:\, | \, x_F^*(z-x) \, | \geq \epsilon \,\right) .\end{array} \tag{$*$}$$

We shall use property $(*)$ to construct an increasing sequence $\{\pi_n\}_{n\geq 0}$ of finite partitions of $[0,1)$, a quasi-martingale $\{f_n, \sigma(\pi_n)\}_{n\geq 0}$ corresponding to $\{\epsilon_n\}$, and a sequence $\{x_n^*\}_{n\geq 1}$ in $S(\mathfrak{X}^*)$ such that for each nonnegative integer n

(1) f_n has the form $f_n = \sum_{E\in\pi_n} x_E \chi_E$ where x_E is in D,

(2) $| \, x_{n+1}^*(f_{n+1} - f_n) \, | \geq \epsilon$ a.e. ,

(3) if E is in π_n, then E has the form $[a,b)$ and $\mu(E) < 1/2^n$ and

(4) $\pi_0 = \{\Omega\}$.

Condition (3) guarantees that $\vee\, \sigma(\pi_n) = \Sigma$ while condition (2) guarantees that $\{f_n\}$ is not Cauchy in the Pettis norm.

Towards the construction, pick an arbitrary x in D. Set $\pi_0 = \{\Omega\}$ and $f_0 = x\,\chi_\Omega$.

Fix $n \geq 0$. Suppose that a partition π_n of Ω into intervals of length at most $1/2^n$ and $f_n = \sum_{E\in\pi_n} x_E\chi_E$ with $x_E \in D$ have been defined. We now construct f_{n+1}, π_{n+1} and x_{n+1}^* satisfying conditions (1), (2) and (3).

To this end, apply $(*)$ to $F = \{x_E \,:\, E \in \pi_n\}$ and find the associated $x_F^* = x_{n+1}^*$ in $S(\mathfrak{X}^*)$. Fix an element $E = [a,b)$ of π_n. We first define $f_{n+1}\chi_E$. Property $(*)$ gives that for a suitable choice of $x_1, \ldots, x_m$ in D and positive real numbers $\alpha_1, \ldots, \alpha_m$ whose sum is one,

$$\| \, x_E \; - \; \sum_{i=1}^m \alpha_i x_i \, \| \leq \; \epsilon_n \qquad \text{and for each } \; i \qquad | \, x_{n+1}^*(x_i - x_E) \, | \geq \; \epsilon \, . \tag{$**$}$$

Using repetition, we arrange to have $\alpha_i < 1/2^{n+1}$ for each i. It follows that there are real numbers $d_0, d_1, \ldots, d_m$ such that

$$a = d_0 < d_1 < \ldots < d_{m-1} < d_m = b$$

and

$$d_i - d_{i-1} = \alpha_i \ (b-a) \quad \text{for} \quad i = 1, \dots, m \quad .$$

Set

$$f_{n+1}\chi_E \ = \ \sum_{i=1}^{m} \ x_i \ \chi_{[d_{i-1}, d_i)} \ .$$

Define f_{n+1} on all of Ω similarly. Let π_{n+1} be the partition consisting of all the intervals $[d_{i-1}, d_i)$ obtained from letting E range over π_n.

Clearly, f_{n+1}, π_{n+1} and x^*_{n+1} satisfy conditions (1), (2) and (3).

Towards insuring that $\{f_n\}$ is indeed a quasi-martingale corresponding to $\{\epsilon_n\}$, fix an $E = [a,b)$ in π_n. Using the above notation, we have that for almost all t in E, $f_n(t) \ = \ x_E$ and

$$\begin{aligned} E_n(f_{n+1})(t) \ &= \ \frac{1}{b-a} \int_a^b f_{n+1} \, d\mu \\ &= \ \frac{1}{b-a} \sum_{i=1}^{m} \int_{d_{i-1}}^{d_i} f_{n+1} \, d\mu \\ &= \ \sum_{i=1}^{m} \ \frac{d_i - d_{i-1}}{b-a} \ x_i \\ &= \ \sum_{i=1}^{m} \ \alpha_i \, x_i \ . \end{aligned}$$

An appeal to condition $(**)$ yields the necessary inequality,

$$\| \ E_n(f_{n+1}) \ - \ f_n \ \|_{L_1} \leq \ \epsilon_n \ .$$

This completes the necessary constructions. $\square$

Remark 3.15. For a convex set D, Bocce dentability and midpoint Bocce dentability coincide since, for a fixed $x^* \in S(\mathfrak{X}^*)$, if $x \in D$ satisfies the condition

$$\text{if} \quad x = \tfrac{1}{2} z_1 + \tfrac{1}{2} z_2 \quad \text{with } z_i \in D \quad \text{then} \quad \mid x^*(x - z_1) \mid \equiv \mid x^*(x - z_2) \mid < \epsilon,$$

then $x \in D$ also satisfies the condition

$$\text{if } x = \sum_{i=1}^{n} \alpha_i z_i \text{ with } z_i \in D, \ \ 0 \leq \alpha_i, \text{ and } \sum_{i=1}^{n} \alpha_i = 1 \ ,$$

$$\text{then } \sum_{i=1}^{n} \alpha_i \,|\, x^*(x - z_i) \,|\, < \; 2\epsilon.$$

Remark 3.16. It is easy to verify that $B(L_1)$ is not weak-norm-one dentable, Bocce dentable, nor midpoint Bocce dentability but that

$$B^+(L_1) \;\equiv\; \{f \in L_1 \;:\| \; f \;\|_{L_1} \leq 1 \text{ and } f \geq 0 \text{ a.e. } \}$$

is weak-norm-one dentable, Bocce dentable, and midpoint Bocce dentability.

Remark 3.17. For a bounded subset D and $\epsilon > 0$, a point $x \in D$ is called an <u>ϵ-strong extreme point</u> of D if there is a $\delta > 0$ such that if x_1, x_2 belong to D and there is a point u on the line segment joining x_1 and x_2 with $\| \; u - x \; \| < \delta$, then $\| \; u - x_1 \; \| < \epsilon$ or $\| \; u - x_2 \; \| < \epsilon$. A closed bounded convex set has the <u>Approximate Krein-Milman property</u> (AKMP) if each of its nonempty subsets has an ϵ-strong extreme point for every $\epsilon > 0$. It is clear from the definitions that if a closed bounded convex set has the AKMP, then it is midpoint Bocce dentable.

Remark 3.18. Analogous to the RNP case, in Theorem 3.8 we would like to reduce our test sets from bounded sets to closed bounded convex sets. However, the author is not certain whether this is possible. A close look at the proofs reveal that we only need that, for each bounded linear operator T from L_1 into $\mathfrak{X}$, the sets of the form

$$\Delta_B \;\equiv\; \left\{ \frac{T(\chi_A)}{\mu(A)} \;:\; A \subset B \text{ and } A \in \Sigma^+ \right\} \quad \text{where } B \in \Sigma^+$$

to be midpoint Bocce dentable. But such sets need not be closed nor convex.

If T is the identity operator on L_1, then Δ_Ω is closed but is not convex.

If $T \,:\, L_1 \to L_2$ is given by $(Tf)(\cdot) \equiv \int_0^{\cdot} f \, d\mu$, then Δ_Ω is neither closed nor convex.

If T is representable by a simple function f and $B \in \Sigma^+$, then Δ_B is the convex hull of the values that f takes on B. Thus, if T is a bounded linear operator into a Banach space with the RNP and $B \in \Sigma^+$, then the $\overline{\Delta_B}$ is convex.

If T is a bounded linear operator into a Banach space with the CCP and $B \in \Sigma^+$, then we have already seen that Δ_B need not be closed nor convex; however, $\overline{\Delta_B}$ is convex. This follows from the fact that for each $\epsilon > 0$ there is an operator T_ϵ (Fact 1.1.8) that is representable by a simple function such that

$$\sup_{f \in B(L_\infty)} \| (T - T_\epsilon)(f) \|_{\mathfrak{X}} < \epsilon .$$

CHAPTER 4

BUSHES AND TREES

In this chapter, we examine which Banach spaces allow certain types of bushes and trees to grow in them. First let us review some known implications.

A Banach space $\mathfrak{X}$ fails the RNP precisely when a bounded δ-bush grows in $\mathfrak{X}$. Thus if a bounded δ-tree grows in $\mathfrak{X}$ then $\mathfrak{X}$ fails the RNP. The converse is false; the Bourgain-Rosenthal space [BR] fails the RNP yet has no bounded δ-trees. However, if $\mathfrak{X}$ is a dual space then the converse does hold.

Bourgain [B2] showed that if $\mathfrak{X}$ fails the CCP then a bounded δ-tree grows in $\mathfrak{X}$. The converse is false; the dual of the James Tree space has a bounded δ-tree and the CCP. It is well-known that if a bounded δ-Rademacher tree grows in $\mathfrak{X}$ then $\mathfrak{X}$ fails the CCP. Riddle and Uhl [RU] showed that the converse holds in a dual space. This chapter's main theorem, Theorem 4.1 below, makes precise exactly which types of bushes and trees grow in a Banach space failing the CCP.

Theorem 4.1. *The following statements are equivalent.*

(1) $\mathfrak{X}$ *fails the CCP.*

(2) *A bounded separated δ-tree grows in $\mathfrak{X}$.*

(3) *A bounded separated δ-bush grows in $\mathfrak{X}$.*

(4) *A bounded δ-Rademacher tree grows in $\mathfrak{X}$.*

That (1) implies (2) will follow from Theorem 4.2 below. All the other implications are straightforward and will be verified shortly. As usual, we start with definitions.

While typing this thesis, I learned that H.P. Rosenthal has also recently obtained the result that if $\mathfrak{X}$ fails the CCP then a bounded δ-Rademacher tree grows in $\mathfrak{X}$.

Perhaps it is easiest to define a bush via martingales. If $\{\pi_n\}_{n\geq 0}$ is an increasing sequence of finite positive interval partitions of $[0,1)$ with $\vee\ \sigma(\pi_n) = \Sigma$ and $\pi_0 = \{\Omega\}$ and if $\{f_n, \sigma(\pi_n)\}_{n\geq 0}$ is an $\mathfrak{X}$-valued martingale, then each f_n has the form

$$f_n = \sum_{E\in\pi_n} x_E^n \chi_E$$

and the system

$$\{x_E^n \,:\, n = 0,1,2,\ldots \quad \text{and} \quad E \in \pi_n\}$$

is a bush in $\mathfrak{X}$. Moreover, every bush is realized this way. A bush is a <u>δ-bush</u> if the corresponding martingale satisfies for each positive integer n

$$\|\, f_n(t) - f_{n-1}(t)\, \| > \delta\ . \tag{i}$$

A bush is a <u>separated δ-bush</u> if there exists a sequence $\{x_n^*\}_{n\geq 1}$ in $S(\mathfrak{X}^*)$ such that the corresponding martingale satisfies for each positive integer n

$$|\ x_n^*\ (f_n(t) - f_{n-1}(t))\ | > \delta\ . \tag{ii}$$

In this case, we say that the bush is separated by $\{x_n^*\}$. Clearly a separated δ-bush is also a δ-bush.

Observation that (3) implies (1) in Theorem 4.1.

If a bounded separated δ-bush grows in a subset D of $\mathfrak{X}$, then condition (ii) guarantees that the corresponding D-valued martingale $\{f_n, \sigma(\pi_n)\}$ is not Pettis-Cauchy since

$$\|\ f_n - f_{n-1}\ \|_{Pettis} \geq \int_\Omega |\ x_n^*\ (f_n(t) - f_{n-1}(t))\ |\ d\mu\ >\ \delta\ .$$

Thus, if a bounded separated δ-bush grows in $\mathfrak{X}$ then $\mathfrak{X}$ fails the CCP (Fact 1.3). $\square$

If each π_n is the n^{th} dyadic partition then we call the bush a (dyadic) tree. Let us rephrase the above definitions for this case, without the help of martingales. A <u>tree</u>

in $\mathfrak{X}$ is a system of the form $\{x_k^n : n = 0, 1, \dots ; \ k = 1, \dots, 2^n\}$ satisfying for $n = 1, 2, \dots$ and $k = 1, \dots, 2^{n-1}$

$$x_k^{n-1} = \frac{x_{2k-1}^n + x_{2k}^n}{2} \ . \tag{iii}$$

Condition (iii) guarantees that $\{f_n\}$ is indeed a martingale. It is often helpful to think of a tree diagrammatically:

$$\begin{array}{cccccccc}
 & & & x_1^0 & & & & \\
 & & x_1^1 & & & x_2^1 & & \\
 & x_1^2 & & x_2^2 & x_3^2 & & x_4^2 & \\
x_1^3 & x_2^3 & x_3^3 & x_4^3 & x_5^3 & x_6^3 & x_7^3 & x_8^3 \\
 & & & \dots & & & &
\end{array}$$

It is easy to see that (iii) is equivalent to

$$x_{2k-1}^n - x_{2k}^n \ = \ 2 \ (\, x_{2k-1}^n - x_k^{n-1} \,) \ = \ 2 \ (\, x_k^{n-1} - x_{2k}^n \,) \ . \tag{iii'}$$

A tree $\{x_k^n\}$ is a <u>δ-tree</u> if for $n = 1, 2, \dots$ and $k = 1, \dots, 2^{n-1}$

$$\| \, x_{2k-1}^n - x_k^{n-1} \, \| \equiv \| \, x_{2k}^n - x_k^{n-1} \, \| > \ \delta \ . \tag{iv}$$

An appeal to (iii') shows that (iv) is equivalent to

$$\| \, x_{2k-1}^n - x_{2k}^n \, \| > 2 \ \delta \ . \tag{iv'}$$

A tree $\{x_k^n\}$ is a <u>separated δ-tree</u> if there exists a sequence $\{x_n^*\}_{n \geq 1}$ in $S(\mathfrak{X}^*)$ such that for $n = 1, 2, \dots$ and $k = 1, \dots, 2^{n-1}$

$$| \, x_n^*(x_{2k-1}^n - x_k^{n-1}) \, | \equiv | \, x_n^*(x_{2k}^n - x_k^{n-1}) \, | > \ \delta \ . \tag{v}$$

Another appeal to (iii') shows that (v) is equivalent to

$$| \, x_n^* (x_{2k-1}^n - x_{2k}^n) \, | > 2 \ \delta \ . \tag{v'}$$

Furthermore, by switching x_{2k-1}^n and x_{2k}^n when necessary, we may assume that (v') is equivalent to

$$x_n^* (x_{2k-1}^n - x_{2k}^n) \ > \ 2 \ \delta \ . \tag{v''}$$

Since a separated δ-tree is also a separated δ-bush, (2) implies (3) in Theorem 4.1.

A tree $\{x_k^n : n = 0, 1, \dots \ ; \ k = 1, \dots , 2^n\}$ is called a <u>δ-Rademacher tree</u> [RU] if for each positive integer n

$$\| \sum_{k=1}^{2^{n-1}} \ (x_{2k-1}^n - x_{2k}^n) \, \| > \ 2^n \delta \ .$$

Perhaps a short word on the connection between the Rademacher functions $\{r_n\}$ and Rademacher trees is in order. In light of our discussion in Chapter 1, there is a one-to-one correspondence between all bounded trees in $\mathfrak{X}$ and all bounded linear operators from L_1 into $\mathfrak{X}$. If $\{x_k^n\}$ is a bounded tree in $\mathfrak{X}$ with associated operator T, then it is easy to verify that $\{x_k^n\}$ is a δ-Rademacher tree precisely when $\| \, T(r_n) \, \| > \delta$ for all positive integers n.

Fact that (4) implies (1) in Theorem 4.1 [RU].

Let $\{f_n\}$ be the (dyadic) martingale associated with a δ-Rademacher tree $\{x_k^n\}$. If x^* is in $\mathfrak{X}^*$ and I_k^n is the dyadic interval $[\ (k-1)/2^n \ , \ k/2^n \ )$ then

$$\begin{aligned}
\int_\Omega | \, x^* \ (f_n - f_{n-1}) \, | \, d\mu \quad &= \quad \sum_{k=1}^{2^{n-1}} \int_{I_k^{n-1}} | \, x^* \ (f_n - f_{n-1}) \, | \, d\mu \\
&= \sum_{k=1}^{2^{n-1}} \left[\int_{I_{2k-1}^n} | \, x^* \ (x_{2k-1}^n - x_k^{n-1}) \, | \, d\mu \ + \ \int_{I_{2k}^n} | \, x^* \ (x_{2k}^n - x_k^{n-1}) \, | \, d\mu \right] \\
&= \frac{1}{2^n} \sum_{k=1}^{2^{n-1}} \left[\, | \, x^* \ (x_{2k-1}^n - x_k^{n-1}) \, | \ + \ | \, x^* \ (x_{2k}^n - x_k^{n-1}) \, | \, \right]
\end{aligned}$$

$$= \frac{1}{2^n} \sum_{k=1}^{2^{n-1}} \mid x^* \ (x^n_{2k-1} - x^n_{2k}) \mid \qquad \text{by (iii')}$$

$$\geq \frac{1}{2^n} \mid x^* \left(\sum_{k=1}^{2^{n-1}} (x^n_{2k-1} - x^n_{2k}) \right) \mid \ .$$

From this we see that $\{f_n\}$ is not Cauchy in the Pettis norm since

$$\begin{aligned} \| f_n - f_{n-1} \|_{Pettis} = & \sup_{x^* \in B(\mathfrak{X}^*)} \int_\Omega \mid x^* \ (f_n - f_{n-1}) \mid d\mu \\ \geq & \sup_{x^* \in B(\mathfrak{X}^*)} \frac{1}{2^n} \mid x^* \left(\sum_{k=1}^{2^{n-1}} (x^n_{2k-1} - x^n_{2k}) \right) \mid \\ = & \frac{1}{2^n} \| \sum_{k=1}^{2^{n-1}} (x^n_{2k-1} - x^n_{2k}) \| \\ > & \frac{1}{2^n} \ 2^n \ \delta \ = \delta \ . \end{aligned}$$

Thus if a bounded δ-Rademacher tree grows in a subset D of $\mathfrak{X}$, then there is a bounded D-valued martingale that in not Pettis-Cauchy and so $\mathfrak{X}$ fails the CCP (Fact 1.3). □

Observation that (2) implies (4) in Theorem 4.1.

A separated δ-tree can easily be reshuffled so that it is a δ-Rademacher tree. For if $\{x^n_k\}$ is a separated δ-tree then we may assume, by switching x^n_{2k-1} and x^n_{2k} when necessary, that there is a sequence $\{x^*_n\}$ in $S(\mathfrak{X}^*)$ satisfying

$$x^*_n(x^n_{2k-1} - x^n_{2k}) \ > \ 2 \ \delta \ .$$

With this modification $\{x^n_k\}$ is a δ-Rademacher tree since

$$\begin{aligned} \| \sum_{k=1}^{2^{n-1}} (x^n_{2k-1} - x^n_{2k}) \| \geq \mid \sum_{k=1}^{2^{n-1}} \ x^*_n \ (x^n_{2k-1} - x^n_{2k}) \mid = & \sum_{k=1}^{2^{n-1}} \ x^*_n \ (x^n_{2k-1} - x^n_{2k}) \\ > & \sum_{k=1}^{2^{n-1}} \ 2 \ \delta \ = \ 2^n \ \delta \ . \qquad \square \end{aligned}$$

To complete the proof of Theorem 4.1, we need only to show that (1) implies (2). Towards this end, let $\mathfrak{X}$ fail the CCP. An appeal to Theorem 3.8 gives that there is a bounded non-midpoint-Bocce-dentable subset of $\mathfrak{X}$. In such a set, we can construct a separated δ-tree. This construction is made precise in the following theorem.

Theorem 4.2. *A separated δ-tree grows in a non-midpoint-Bocce-dentable set.*

Proof. Let D be a subset of $\mathfrak{X}$ that is not midpoint Bocce dentable. Accordingly, there is a $\delta > 0$ satisfying:

$$\begin{gathered}\text{for each finite subset } F \text{ of } D \text{ there is } \; x_F^* \in S(\mathfrak{X}^*) \; \text{ such that} \\ \text{each } x \text{ in } F \text{ has the form} \quad x = \frac{x_1 + x_2}{2} \qquad \text{with} \qquad |x_F^*(x_1 - x_2)| > \delta \qquad (*) \\ \text{for a suitable choice of } x_1 \text{ and } x_2 \text{ in } D\,.\end{gathered}$$

We shall use the property $(*)$ to construct a tree $\{x_k^n : n = 0, 1, \ldots ; k = 1, \ldots, 2^n\}$ in D that is separated by a sequence $\{x_n^*\}_{n=1}$ of norm one linear functionals.

Towards this construction, let x_1^0 be any element of D. Apply $(*)$ with $F = \{x_1^0\}$ and find $x_F^* = x_1^*$. Property $(*)$ provides x_1^1 and x_2^1 in D satisfying

$$x_1^0 = \frac{x_1^1 + x_2^1}{2} \qquad \text{and} \qquad |x_1^*(x_1^1 - x_2^1)| > \delta\,.$$

Next apply $(*)$ with $F = \{x_1^1, x_2^1\}$ and find $x_F^* = x_2^*$. For $k = 1$ and 2, property $(*)$ provides x_{2k-1}^2 and x_{2k}^2 in D satisfying

$$x_k^1 = \frac{x_{2k-1}^2 + x_{2k}^2}{2} \qquad \text{and} \qquad |x_2^*(x_{2k-1}^2 - x_{2k}^2)| > \delta\,.$$

Instead of giving a formal inductive proof we shall be satisfied by finding x_3^* along with $x_1^3, x_2^3, \ldots, x_8^3$ in D. Apply $(*)$ with $F = \{x_1^2, x_2^2, x_3^2, x_4^2\}$ and find $x_F^* = x_3^*$. For $k = 1, 2, 3$ and 4, property $(*)$ provides x_{2k-1}^3 and x_{2k}^3 in D satisfying

$$x_k^2 = \tfrac{1}{2}(x_{2k-1}^3 + x_{2k}^3) \qquad \text{and} \qquad |x_3^*(x_{2k-1}^3 - x_{2k}^3)| > \delta\,.$$

It is now clear that a separated δ-tree grows in such a set D. $\square$

Remark 4.3. Theorem 3.8 presents several dentability characterizations of the CCP. Our proof that (1) implies (2) in Theorem 4.1 uses part of one of these characterizations; namely, if $\mathfrak{X}$ fails the CCP then there is a bounded non-midpoint-Bocce-dentable subset of $\mathfrak{X}$. If $\mathfrak{X}$ fails the CCP, then there is also a bounded non-weak-norm-one-dentable subset of $\mathfrak{X}$ (Theorem 3.8). In the closed convex hull of such a set we can construct a martingale that is not Pettis-Cauchy [Fact 3.14]; furthermore, the bush associated with this martingale is a separated δ-bush. However, it is unclear whether this martingale is a dyadic thus the separated δ-bush may not be a tree. If $\mathfrak{X}$ fails the CCP, then there is also a bounded non-Bocce-dentable subset of $\mathfrak{X}$ (Theorem 3.8). In such a set we can construct a martingale that is not Pettis-Cauchy (Theorem 3.9), but it is unclear whether the bush associated with this martingale is a separated δ-bush.

Remark 4.4. The Rademacher functions $\{r_n\}$ may be viewed as a test sequence for the CCP. For if $\mathfrak{X}$ has the CCP and T is a bounded linear operator from L_1 into $\mathfrak{X}$, then $\{T(r_n)\}$ converges to 0 in norm. If $\mathfrak{X}$ fails the CCP, then there is a bounded linear operator T from L_1 into $\mathfrak{X}$ such that $\inf_n \| T(r_n) \| > \delta$ some $\delta > 0$.

Remark 4.5. Recall that a separated δ-tree is also a δ-tree and, after a reshuffling, a δ-Rademacher tree. However, a bounded $\tilde{\delta}$-Rademacher tree need neither be a δ-tree nor a separated δ-tree. For example, consider the c_0-valued dyadic martingale $\{f_n \equiv (s_0, \dots, s_n, 0, 0, \dots)\}$ where the function s_n from $[0,1]$ into $[-1,1]$ is defined by

$$s_n = \begin{cases} (-1)^k \, 2^{-n} & \text{if } \omega \in I_k^n \text{ with } k \le 2 \,; \\ (-1)^k & \text{if } \omega \in I_k^n \text{ with } k > 2 \,. \end{cases}$$

The tree associated with $\{f_n\}$ is a bounded $\frac{1}{4}$-Rademacher tree but is neither a δ-tree nor a separated δ-tree for any positive δ. Thus, since a δ-tree grows in a space failing the CCP, the notion of a separated δ-tree is preferred to that of a δ-Rademacher tree for characterizing the CCP.

Remark 4.6. The δ-tree that Bourgain [B2] constructed in a space failing the CCP is neither a separated δ-tree nor a δ-Rademacher tree since the operator associated with his tree is Dunford-Pettis. Modifying Bourgain's construction, we can produce a bounded δ-Rademacher tree; however, it is not clear that this tree is a δ-tree. We now present this construction.

A direct proof that a bounded δ-Rademacher tree grows in a space failing the CCP.

Let $\mathfrak{X}$ be a Banach space failing the CCP. Fix a non-Dunford-Pettis operator T from L_1 into $\mathfrak{X}$. By a lemma of Bourgain [B2], there exists $A \in \Sigma^+$, $\delta > 0$, and a sequence $\{g_r\}$ of simple functions in L_1 such that

(i) $\| g_r \|_{L_\infty} \leq 1$

(ii) $\{g_r\}$ is weakly null in L_1

(iii) $\overline{\lim} \| T(f \, g_r) \| \geq 3\delta \| f \|_{L_1}$ if $f \in \mathcal{F}(A)$

where $\mathcal{F}(A) \equiv \{h \in L_1 : \| h \|_{L_1} = \| h\chi_A \|_{L_1} , \ h \geq 0 \ \text{a.e.} \}$.

We shall construct, by induction on n, a system

$$\{f_k^n \, : \, n = 0, 1, \ldots \ \text{and} \ k = 1, \ldots , 2^n\}$$

of L_1-functions satisfying for each admissible n and k

(1) f_k^n is a simple function in $\mathcal{F}(A)$

(2) $\| f_k^n \|_{L_1} = 1$

(3) $2 \, f_k^n = f_{2k-1}^{n+1} + f_{2k}^{n+1}$

(4) $\| \sum_{k=1}^{2^n} (T f_{2k-1}^{n+1} - T f_{2k}^{n+1}) \| > \delta 2^{n+1}$.

It is easy to see that $\{T f_k^n\}$ will be a bounded δ-Rademacher tree in $\mathfrak{X}$ (in fact the tree is in the image of the probability densities of L_1 with support in A).

Set $f_1^0 = \frac{\chi_A}{\mu(A)}$. Clearly, f_1^0 satisfies conditions (1) and (2).

Fix $n \geq 0$. Assume that we have constructed a system $\{f_k^n : k = 1, \ldots , 2^n\}$ that satisfies conditions (1) and (2). We shall now construct $\{f_k^{n+1} : k = 1, \ldots , 2^{n+1}\}$ such that the system $\{f_k^m : m = n, n+1 \text{ and } k = 1, \ldots , 2^m\}$ satisfies conditions (1) through (4).

Set $f_n = 2^{-n} \sum_{k=1}^{2^n} f_k^n$. Note that $f_n \in \mathcal{F}(A)$ and $\| f_n \|_{L_1} = 1$. Applying Bourgain's lemma, we may assume (just pass to a subsequence) that for each positive integer r, $\| T(f_n \, g_r) \| \geq 3\delta$.

Let $\mathcal{A}_n$ be the (finite) algebra generated by $\{f_k^n : k = 1, \ldots , 2^n\}$ and $E_n(\cdot)$ be the conditional expectation with respect to $\mathcal{A}_n$. Set

$$g_r' = g_r - E_n(g_r) \ .$$

Since $\{g_r\}$ tends weakly to 0, $\| E_n(g_r) \|_{L_\infty}$ tends to 0. Thus, we may choose a positive integer $R_n \equiv R$ such that

$$\| g_R' \|_{L_\infty} \leq 2 \qquad \text{and} \qquad \| E_n(g_R) \|_{L_\infty} < \frac{\delta}{\| T \|} \ .$$

Note that for $k = 1, \ldots , 2^n$, since each f_k^n is $\mathcal{A}_n$- measurable, $\int_\Omega f_k^n g_R' \, d\mu = 0$.

Set, for $k = 1, \ldots , 2^n$,

$$f_{2k-1}^{n+1} = f_k^n + \tfrac{1}{2} f_k^n g_R' \equiv f_k^n \, (1 + \tfrac{1}{2} g_R')$$

and

$$f_{2k}^{n+1} = f_k^n - \tfrac{1}{2} f_k^n g_R' \equiv f_k^n \, (1 - \tfrac{1}{2} g_R') \ .$$

Clearly, $\{f_k^m : m = n, n+1 \text{ and } k = 1, \ldots , 2^m\}$ satisfies conditions (1), (2), and

(3). As for condition (4),

$$\begin{aligned}
\| \sum_{k=1}^{2^n} (Tf_{2k-1}^{n+1} \; - \; Tf_{2k}^{n+1}) \| &= \| \sum_{k=1}^{2^n} T(f_k^n g_R') \| \\
&= \| \; T(2^n f_n g_R') \; \| \\
&\geq \| \; T(2^n f_n g_R) \; \| \; - \; \| \; T(2^n f_n E_n(g_R)) \; \| \\
&\geq \; 3\delta 2^n \; - \; 2^n \; \| T \| \, \| f_n \|_{L_1} \, \| E_n(g_R) \|_{L_\infty} \\
&> \; 3\delta 2^n \; - \; 2^n \; \delta \\
&= \delta \; 2^{n+1} \; .
\end{aligned}$$

Thus, condition (4) is also satisfied. $\square$

CHAPTER 5

LOCALIZATION

We now localize the results thus far. We define the CCP for bounded subsets of $\mathfrak{X}$ by examining the behavior of certain bounded linear operators from L_1 into $\mathfrak{X}$. Before determining precisely which operators let us set some notation and consider an example.

Let $F(L_1)$ denote the positive face of the unit ball of L_1, i.e.

$$F(L_1) = \{f \in L_1 \, : \, f \geq 0 \ \text{ a.e. } \ \text{and} \quad \| \, f \, \| = 1\} \, ,$$

and Δ denote the subset of $F(L_1)$ given by

$$\Delta = \Big\{ \frac{\chi_E}{\mu(E)} \, : \, E \in \Sigma^+ \Big\} \, .$$

Note that the L_1-norm closed convex hull of Δ is $F(L_1)$.

Some care is needed in localizing the CCP. The example below (due to Stegall) illustrates the trouble one can encounter in localizing the RNP.

Example 5.1. We would like to define the RNP for sets in such a way that if a subset D has the RNP then the $\overline{\text{co}}\, D$ also has the RNP. For now, let us agree that a subset D of $\mathfrak{X}$ has the RNP if all bounded linear operators from L_1 into $\mathfrak{X}$ with $T(\Delta) \subset D$ are representable. Let $\mathfrak{X}$ be any separable Banach space without the RNP (e.g. L_1). Renorm $\mathfrak{X}$ to be a strictly convex Banach space. Let D be the unit sphere of $\mathfrak{X}$ and $T \, : \, L_1 \to \mathfrak{X}$ satisfy $T(\Delta) \subset D$. Since $\mathfrak{X}$ is strictly convex, it is easy to verify that $T(\Delta)$ is a singleton in $\mathfrak{X}$. Thus T is representable and so D has the RNP. If this is to imply that $\overline{\text{co}}\, D$ also has the RNP, then the unit ball of $\mathfrak{X}$ would have the RNP. But if the unit ball of $\mathfrak{X}$ has the RNP then $\mathfrak{X}$ has the RNP; but, $\mathfrak{X}$ fails the

RNP. The same problem arises if we replace $T(\Delta) \subset D$ by either $T(F(L_1)) \subset D$ or $T(B(L_1)) \subset D$.

Because of such difficulties, we localize properties to nonconvex sets by considering their closed convex hull. We now make precise the localized definitions.

Definition 5.2. If D is a closed bounded convex subset of $\mathfrak{X}$, then D has the complete continuity property if all bounded linear operators T from L_1 into $\mathfrak{X}$ satisfying $T(\Delta) \subset D$ are Dunford-Pettis. If D is an arbitrary bounded subset of $\mathfrak{X}$, then D has the complete continuity property if the $\overline{\text{co}}\ D$ has the complete continuity property.

The RNP and strong regularity for subsets are defined similarly. We obtain equivalent formulations of the above definitions by replacing $T(\Delta)$ with $T(F(L_1))$. Because of the definitions we restrict our attention to closed bounded convex subsets of $\mathfrak{X}$.

The oscillation results of Chapter 2 localize easily using Corollary 2.8.

Theorem 5.3. *If K is a closed bounded convex subset of $\mathfrak{X}$, then the following statements are equivalent.*

(1) *K has the CCP.*
(2) *For each bounded linear operator $T : L_1 \to \mathfrak{X}$ with $T(\Delta) \subset K$, the subset $T^*(B(\mathfrak{X}^*))$ of L_1 is a set of small Bocce oscillation.*
(3) *For each bounded linear operator $T : L_1 \to \mathfrak{X}$ with $T(\Delta) \subset K$, the subset $T^*(B(\mathfrak{X}^*))$ of L_1 satisfies the Bocce criterion.*

We can localize the martingale characterization of the CCP. As in Chapter 1, fix an increasing sequence $\{\pi_n\}_{n\geq 0}$ of finite positive interval partitions of Ω such that $\vee\ \sigma(\pi_n) = \Sigma$ and $\pi_0 = \{\Omega\}$. Set $\mathcal{F}_n = \sigma(\pi_n)$. It is easy to see that a martingale

$\{f_n, \mathcal{F}_n\}$ takes values in K precisely when the corresponding operator T satisfies $T(\Delta) \subset K$. In light of Fact 1.3, we now have the following fact.

Fact 5.4. If K is a closed bounded convex subset of $\mathfrak{X}$, then K has the CCP precisely when all K-valued martingales are Cauchy in the Pettis norm.

Theorem 3.8 localizes to provide the following characterization.

Theorem 5.5. *Let K be a closed bounded convex subset of $\mathfrak{X}$. The following statements are equivalent.*

(1) *K has the CCP.*

(2) *All the subsets of K are weak-norm-one dentable.*

(3) *All the subsets of K are midpoint Bocce dentable.*

(4) *All the subsets of K are Bocce dentable.*

Proof. It is clear from the definitions that (2) implies (3) and that (4) implies (3). Theorem 3.9 and Fact 5.4 show that (1) implies (4) while Fact 3.14 and Fact 5.4 show that (1) implies (2). So we only need to show that (3) implies (1). For this, slight modifications in the proof of Theorem 3.11 suffice.

Let all subsets of K be midpoint Bocce dentable. Fix a bounded linear operator T from L_1 into $\mathfrak{X}$ satisfying $T(\Delta) \subset K$. We shall show that the subset $T^*(B(\mathfrak{X}^*))$ of L_1 satisfies the Bocce criterion. Then an appeal to Theorem 5.3 gives that K has the complete continuity property. To this end, fix $\epsilon > 0$ and B in Σ^+. Let F denote the vector measure from Σ into $\mathfrak{X}$ given by $F(E) = T(\chi_E)$. Since $T(\Delta) \subset K$, the set $\{\frac{F(E)}{\mu(E)} : E \subset B \text{ and } E \in \Sigma^+\}$ is a subset of K and thus is midpoint Bocce dentable. The proof now proceeds as the proof of Theorem 3.11. $\square$

Towards a localized tree characterization, let K be a closed bounded convex subset of $\mathfrak{X}$. If K fails the CCP, then there is a subset of K that is not midpoint

Bocce dentable (Theorem 5.5) and hence a separated δ-tree grows in K (Theorem 4.2). A separated δ-tree is a separated δ-bush and, with slight modifications, a δ-Rademacher tree. In light of our discussion in Chapter 4, if a separated δ-bush or a δ-Rademacher tree grows in K, then the associated K-valued martingale is not Pettis-Cauchy and so K fails the CCP (Fact 5.4). Thus Theorem 4.1 localizes to provide the following characterization.

Theorem 5.6. *Let K be a closed bounded convex subset of $\mathfrak{X}$. The following statements are equivalent.*

(1) *K fails the CCP.*

(2) *A separated δ-tree grows in K.*

(3) *A separated δ-bush grows in K.*

(4) *A δ-Rademacher tree grows in K.*

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