Def. 2.1.1. Let X be a non-empty set and $d\colon X \times X \to \mathbb{R}$ be s.t. $\forall x, y, z \in X$:

(1a) $d(x,y) \geq 0$
(1b) $d(x,y) = 0 \quad \Leftrightarrow \quad x = y$
(2) $d(x,y) = d(y,x)$ (i.e., d is *symmetric*)
(3) $d(x,y) \leq d(x,z) + d(z,y)$ (i.e., d satisfies the *triangle inequality*).

Then

- d is called a *metric* (or *distance*) function on X
- (X,d) (or just X if d is understood) is called a *metric space*.

►. Below is a collection of definitions from §2.1. Many of the below concepts have many equivalent formulations and often a book chooses one of equivalent formulations as the definition and then shows that the chosen definition is equivalent to the other formulations.

For the remainder of this handout, let: (X,d) be a metric space , $x \in X$, $S \subset X$, $\varepsilon > 0$.

Def. 2.1.3. The *open ball in* X with center x and radius ε is

$$\underbrace{B(x,\varepsilon)}_{\text{book}} \overset{\text{or}}{=} \underbrace{B_\varepsilon(x) \overset{\text{or}}{=} N_\varepsilon(x)}_{\text{prof. mg}} \overset{\text{def.}}{:=} \{y \in X \colon d(x,y) < \varepsilon\} \ .$$

Def. The *deleted* ε*-open ball* about x is $B'_\varepsilon(x) \overset{\text{or}}{=} N'_\varepsilon(x) \overset{\text{def.}}{:=} B_\varepsilon(x) \setminus \{x\} \overset{\text{note}}{=} \{y \in X \colon 0 < d(x,y) < \varepsilon\}$.

Def. x is an *interior* point of $S \overset{\text{def.}}{\iff} (\exists \varepsilon_x > 0)\ [\ B_{\varepsilon_x}(x) \subset S\ ]$.
Notation. $\operatorname{int} S$ denotes the set of all interior points of S.

Def. 2.1.3. S is *open* in $X \overset{\text{def.}}{\iff} (\forall x \in S)\ [\ x \in \operatorname{int} S\ ]. \overset{\text{note}}{\iff} (\forall x \in S)\ (\exists \varepsilon_x > 0)\ [\ B_{\varepsilon_x}(x) \subset S\ ]$.

Def. 2.1.9. S is *closed* in $X \overset{\text{def.}}{\Leftrightarrow} X \setminus S$ is open in X.

Recall: $S^c \overset{\text{notation}}{=} X \setminus S \overset{\text{def.}}{:=} \{x \in X \colon x \notin S\} \underset{\text{notation}}{\overset{\text{book's}}{=}} {}^cS$.

Def. 2.1.6. *interior of* $S \overset{\text{notation}}{=} S^\circ \overset{\text{def.}}{:=} \bigcup \{G \colon G \subset S$ and G is open in $X\} \underset{\text{notation}}{\overset{\text{book's}}{=}} \overset{\circ}{S}$.

Def. 2.1.12. *closure of* $S \overset{\text{notation}}{=} \overline{S} \overset{\text{def.}}{:=} \bigcap \{F \colon F \supset S$ and F is closed in $X\}$.

Def. 2.1.17. $\partial S \overset{\text{or}}{=}$ boundary of $S \overset{\text{def.}}{:=} \overline{S} \setminus S^\circ$

Def. 2.1.15. S is a *neighborhood* (NBHD) *of* x provided S is an open set that contains x.[1]

Recall. The set of extended real numbers is $\widehat{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$. Also, for $R \subset \mathbb{R}$,

$$\sup \emptyset = \operatorname{lub} \emptyset = -\infty \qquad \text{and} \qquad \sup R = -\infty \quad \Leftrightarrow \quad R = \emptyset$$
$$\inf \emptyset = \operatorname{glb} \emptyset = +\infty \qquad \text{and} \qquad \inf R = +\infty \quad \Leftrightarrow \quad R = \emptyset \ .$$

Def. 2.1.22. bounded (abbreviated bnd)
∘ $\operatorname{diam} S \overset{\text{or}}{=}$ *diameter of* $S := \sup\{d(x,y) : x, y \in S\} \in \widehat{\mathbb{R}}$.
∘ S is *bounded* (bnd) $\overset{\text{def.}}{\Leftrightarrow} \operatorname{diam} S < \infty$.
$\overset{\text{note}}{\Leftrightarrow} (\exists M \in \mathbb{R})\ (\forall x, y \in S)\ [\ d(x,y) \leq M\ ]$. (terminology: S is bounded by M)

Def. 2.1.25. x is a *limit* (or *accumulation*) *point* of $S \overset{\text{def.}}{\Leftrightarrow} \forall$ NBHD V of x , $\exists s \in S \cap [V \setminus \{x\}]$
$\overset{\text{note}}{\Leftrightarrow} \forall \ \varepsilon > 0$, $\exists s \in S \cap [B_\varepsilon(x) \setminus \{x\}]$
$\overset{\text{note}}{\Leftrightarrow} \forall \ \varepsilon > 0$, $\exists s \in S \cap B'_\varepsilon(x)$

Notation. S' denotes the set of all limit points of S.

[1] think of $B_\varepsilon(x)$ as a *basic* NBHD of x

INTERIOR OF A SUBSET S OF A METRIC SPACE X

(1i) interior of S $\overset{\text{notation}}{=}$ S^o $\overset{\text{def}}{=}$ $\bigcup_{G \in \mathcal{G}_S} G$ where $\mathcal{G}_S = \{G \in \mathcal{P}(X) : G \text{ is open and } G \subset S\}$.

(2i) S^o is open

(3i) $S^o \subset S$

(4i) S^o is the largest open set contained in S $\langle$ vaguely, S^o is the largest open set *inside of* S $\rangle$

in the sense that S^o is an open set contained in S and $\langle$ now largest part $\rangle$ if H is an open set contained in S then $H \subset S^o$.

(5i) $S^o = S \iff S$ is open

(6i) $S^o = \operatorname{int} S$ $\langle$ i.e., the interior of S equals the set of interior points of S $\rangle$

(7i) $(S^o)^c = \overline{S^c}$

(8i) $x \in S^o \Leftrightarrow (\exists \varepsilon_x > 0)\,[\, B_{\varepsilon_x}(x) \subset S \,]$

CLOSURE OF A SUBSET S OF A METRIC SPACE X

(1c) closure of S $\overset{\text{notation}}{=}$ $\overline{S}$ $\overset{\text{def}}{=}$ $\bigcap_{F \in \mathcal{F}_S} F$ where $\mathcal{F}_S = \{F \in \mathcal{P}(X) : F \text{ is closed and } S \subset F\}$.

(2c) $\overline{S}$ is closed

(3c) $S \subset \overline{S}$

(4c) $\overline{S}$ is the smallest closed set that contains S $\langle$ vaguely, $\overline{S}$ is the smallest closed set that *sits on top of* S $\rangle$

in the sense that $\overline{S}$ is a closed set that contains S and $\langle$ now smallest part $\rangle$ if H is an closed set that contains S then $\overline{S} \subset H$.

(5c) $S = \overline{S} \iff S$ is closed

(6c) $\overline{S} = S \cup S' = S \cup \partial S$

(7c) $\left(\overline{S}\right)^c = (S^c)^o$

(8c) $x \in \overline{S} \Leftrightarrow (\forall \varepsilon > 0)\,[\, B_\varepsilon(x) \cap S \neq \emptyset \,]$