**On this page**

- $X$ is a nonempty set,
- $S$ is a nonempty subset of of metric space,
- $I$ is a nonempty closed and bounded interval of $\mathbb{R}$,
- $\mathbb{K}$ is the field of either $\mathbb{R}$ or $\mathbb{C}$.

## Function Spaces

Consider the following spaces of functions (i.e., Function Spaces).

$$\mathscr{B}(X,\mathbb{K}) := \{f\colon X \to \mathbb{K} \mid f \text{ is bounded } X\} \tag{1}$$

$$\mathscr{C}(S,\mathbb{K}) := \{f\colon S \to \mathbb{K} \mid f \text{ is bounded and continuous on } S\} \tag{2}$$

$$C(S,\mathbb{K}) := \{f\colon S \to \mathbb{K} \mid f \text{ is continuous on } S\} \tag{3}$$

$$\mathscr{R}(I,\mathbb{K}) := \{f\colon I \to \mathbb{K} \mid f \text{ is Riemann Integrable over } I\} \tag{4}$$

and for $n \in \mathbb{N} \cup \{0\}$ $\langle$where $f^{(n)}$ is the $n^{\text{th}}$ deriviative of $f$ with $f^{(0)} := f$ and so $C^0(I,\mathbb{K}) = C(I,\mathbb{K})\rangle$

$$C^n(I,\mathbb{K}) := \{f\colon I \to \mathbb{K} \mid f \text{ is } n\text{-times continuously differentiable on } I\} \tag{5}$$

$$\overset{\text{i.e.}}{=} \left\{f\colon I \to \mathbb{K} \mid f^{(n)} \text{ is continuous of } I\right\}$$

Note $C^0(I,\mathbb{R}) = C(I,\mathbb{R}) = \mathscr{C}(I,\mathbb{R}) \subset \mathscr{B}(I,\mathbb{R})$ while $C(I,\mathbb{R}) \subset \mathscr{R}(I,\mathbb{R}) \subset \mathscr{B}(I,\mathbb{R})$.

The function space $\mathscr{B}(X,\mathbb{K})$ is also denoted by $\mathscr{B}_{\mathbb{K}}(X)$. Similary with the other function spaces.

If the field $\mathbb{K}$ is understood by context, then the dependancy on $\mathbb{K}$ is often omitted in the notation.

## Metrics on Functions Spaces with $\mathbb{K} = \mathbb{R}$

The usual metric[1] on $\mathscr{B}(X)$ is

$$d_\infty(f,g) := \sup_{x \in X} |f(x) - g(x)|. \tag{6}$$

Since $C(I) \subset \mathscr{B}(I)$, we get $d_\infty\big|_{C(I)\times C(I)}$ is a metric on $C(I)$, which we often (abusively) denote by just $d_\infty$. Similarly for the other subsets of $\mathscr{B}(X)$.

Let $1 \le p \le \infty$. Other metrics[2] on $C(I)$ are

$$d_p(f,g) := \begin{cases} \left[\int_I |f(t) - g(t)|^p \, dt\right]^{1/p} & , 1 \le p < \infty \\ \max\limits_{t \in I} |f(t) - g(t)| & , p = \infty. \end{cases} \tag{7}$$

On $C(I)$, the $d_\infty$ as given by (7) equals the $d_\infty$ as given by (6), which is the usual metric on $C(I)$.

A common metric[3] on $C^n(I)$ is

$$\rho_\infty(f,g) := \sum_{k=0}^{n} \max_{t \in I} \left|f^{(k)}(t) - g^{(k)}(t)\right| \underset{d_\infty \text{ from (7)}}{\overset{\text{i.e., using the}}{=}} \sum_{k=0}^{n} d_\infty\left(f^{(k)}, g^{(k)}\right). \tag{8}$$

If $n = 0$, then $C^0(I) = C(I)$ and the $\rho_\infty$ as given in (8) equals the $d_\infty$ as given in (7).

[1] see p. 71 (vii), whose proof easily generalizes from $I$ to any nonempty set $X$
[2] see p. 70 (v) and p. 90 Execise 2.1.45.2
[3] straight forward to check since the $d_\infty$ from (7) is a metric

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Completeness

(1) **Ex 2.2.2(v)** The metric space $(C(I)\, d_p)$ is not complete for $1 \leq p < \infty$.

(2) **Thm 2.2.5**. The metric space $(\mathscr{B}(X), d_\infty)$ is complete.

(3) **Thm 2.2.6**. The metric space $(\mathscr{C}(S), d_\infty)$ is complete.

(4) **Cor. 2.2.7**. The metric space $(C(I), d_\infty)$ is complete.

(5) **Thm 2.2.8**. The metric space $(\mathscr{R}(I), d_\infty)$ is complete.

(6) **Thm 2.2.9** The metric space $(C^n(I), \rho_\infty)$ is complete when $n \in \mathbb{N}$.

Remarks:

The proof on p. 94 of (1) for when $p = 1$ easily generalized to $1 \leq p < \infty$.

The proof of (2) – (5) is on p. 97–98.

The proof on p. 98 of (6) for when $n = 1$ easily generalizes to $n \in \mathbb{N}$.