§2B Measurable Spaces and Functions

Towards our goal "nice" Σ (in class) $\overset{\text{or}}{=}$ $\mathcal{S}$ (in book) $\subseteq \mathcal{P}(\mathbb{R})$ ← power set, the collection of all subsets of $\mathbb{R}$.

$\mathcal{S}$ ← a collection of some subsets of $\mathbb{R}$

2.23 Definition ***σ-algebra***

Suppose X is a set and $\mathcal{S}$ is a set of subsets of X. Then $\mathcal{S}$ is called a *σ-algebra* on X if the following three conditions are satisfied:

- (1) $\varnothing \in \mathcal{S}$;
- (2) if $E \in \mathcal{S}$, then $X \setminus E \in \mathcal{S}$;
- (3) if $E_1, E_2, \ldots$ is a sequence of elements of $\mathcal{S}$, then $\bigcup_{k=1}^{\infty} E_k \in \mathcal{S}$.

Note (1) can be replaced by
(1') Σ is nonempty

2.25 ***σ-algebras are closed under countable intersection***

Suppose $\mathcal{S}$ is a σ-algebra on a set X. Then

(a) $X \in \mathcal{S}$;

(b) if $D, E \in \mathcal{S}$, then $D \cup E \in \mathcal{S}$ and $D \cap E \in \mathcal{S}$ and $D \setminus E \in \mathcal{S}$;

(c) if $E_1, E_2, \ldots$ is a sequence of elements of $\mathcal{S}$, then $\bigcap_{k=1}^{\infty} E_k \in \mathcal{S}$.

Why?

(a) $X = X \setminus \overset{\Sigma}{\overset{\in}{\varnothing}} \overset{\text{so by (2)}}{\in} \Sigma$

(b) $D \cup E \in \Sigma$ b/c Σ closed ctb. Union and $\varnothing \in \Sigma$

$D \cap E = \underset{\Sigma}{\underset{\in}{D^{c}}} \overset{\text{by (2)}}{\cup} \underset{\Sigma}{\underset{\in}{E^{c}}}) \in \Sigma$

$D \setminus E = \overset{\in}{D} \cap \overset{\in}{X \setminus E} \in \Sigma$

(c) $\bigcap E_k = \left(\bigcup E_k^c\right)^c$

Note $\mathcal{S} \subseteq \mathcal{P}(X)$.

2.26 Definition ***measurable space; measurable set*** (✱)

- A *measurable space* is an ordered pair $(X, \mathcal{S})$, where X is a set and $\mathcal{S}$ is a σ-algebra on X.
- An element of $\mathcal{S}$ is called an $\mathcal{S}$*-measurable set*, or just a *measurable set* if $\mathcal{S}$ is clear from the context.

2.27 ***smallest σ-algebra containing a collection of subsets***

Suppose X is a set and $\mathcal{A}$ is a set of subsets of X. Then the intersection of all σ-algebras on X that contain $\mathcal{A}$ is a σ-algebra on X, denoted by $\sigma(\mathcal{A})$

← So $\mathcal{A} \subset \mathcal{P}(X)$.

Rmk

Let $\mathcal{G}_{\mathcal{A}} := \{\Sigma \subset \mathcal{P}(X) : \mathcal{A} \subset \Sigma$ and Σ is a σ-algebra on $X\}$.

So $\sigma(\mathcal{A}) := \bigcap \{\Sigma : \underset{①}{\mathcal{A} \subset \Sigma}$ and Σ is σ-alg. on $X\} \overset{\text{or}}{=} \bigcap_{\Sigma \in \mathcal{G}_{\mathcal{A}}} \Sigma$

So $\mathcal{A} \overset{③\text{ note}}{=} \bigcap_{\Sigma \in \mathcal{G}_{\mathcal{A}}} \mathcal{A} \subset \bigcap_{\Sigma \in \mathcal{G}_{\mathcal{A}}} \Sigma \overset{②\text{ note}}{:=} \sigma(\mathcal{A})$

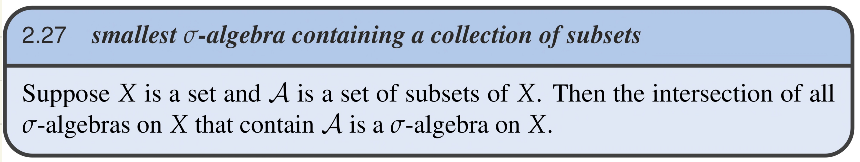
So $\boxed{\mathcal{A} \subseteq \sigma(\mathcal{A}) \subseteq \mathcal{P}(X)}$

$\sigma(\mathcal{A})$ is called the σ-algebra generated by $\mathcal{A}$.

By 2.27, $\sigma(\mathcal{A})$ is the smallest σ-algebra that contains $\mathcal{A}$.

2B2

Proof of



2.27 ***smallest σ-algebra containing a collection of subsets***

Suppose X is a set and $\mathcal{A}$ is a set of subsets of X. Then the intersection of all σ-algebras on X that contain $\mathcal{A}$ is a σ-algebra on X.

LTGBG and $\mathcal{G}_{\mathcal{A}} := \{\Sigma : \Sigma \supseteq \mathcal{A}$ and Σ is a σ-alg. on $X\}$,

WTS $\sigma(\mathcal{A}) := \bigcap_{\Sigma \in \mathcal{G}_{\mathcal{A}}} \Sigma$ is a σ-alg.

(1) Claim $\emptyset \in \sigma(\mathcal{A})$

[$\forall \Sigma \in \mathcal{G}_{\mathcal{A}}$, $\emptyset \in \Sigma$ <b/c Σ is a σ-alg>.

(2) claim $\sigma(\mathcal{A})$ is closed under complementation

Fix $E \in \sigma(\mathcal{A})$; recall $\bigcap_{\Sigma \in \mathcal{G}_{\mathcal{A}}} \Sigma$.

So $\forall \Sigma \in \mathcal{G}_{\mathcal{A}}$: $E \in \Sigma$, so b/c Σ is σ-alg, $X \setminus E \in \Sigma$.

So $X \setminus E \in \bigcap_{\Sigma \in \mathcal{G}_{\mathcal{A}}} \Sigma := \sigma(\mathcal{A})$.

(3) Claim $\sigma(\mathcal{A})$ is closed under ctb. unions

Fix $\{E_k\}_{k=1}^{\infty}$ from $\sigma(\mathcal{A})$. <note already show $\emptyset \in \sigma(\mathcal{A})$ so, wlog, ∞ seq.>

So for each $k \in \mathbb{N}$ and for each $\Sigma \in \mathcal{G}_{\mathcal{A}}$, $E_k \in \Sigma$.

$\xRightarrow{\Sigma \text{ is closed under ctb. unions}} \bigcup_{k=1}^{\infty} E_k \in \Sigma, \forall \Sigma \in \mathcal{G}_{\mathcal{A}}$.

$\xRightarrow{\text{def of } \sigma(\mathcal{A})} \bigcup_{k=1}^{\infty} E_k \in \sigma(\mathcal{A})$. ∎

Cor. to 2.27 Let $\mathcal{E} \subset \mathcal{P}(X)$ for some set X.

Then $\sigma(\sigma(\mathcal{E})) = \sigma(\mathcal{E})$.

why? $\underbrace{\sigma(\sigma(\mathcal{E}))}_{\text{is the smallest } \sigma\text{-alg. that contains } \underbrace{\sigma(\mathcal{E})}_{\text{this is a } \sigma\text{-alg}}}$.

2B3

2.29 Definition *Borel set*

The smallest σ-algebra on **R** containing all open subsets of **R** is called the collection of *Borel subsets* of **R**. An element of this σ-algebra is called a *Borel set*.

Common Notation:

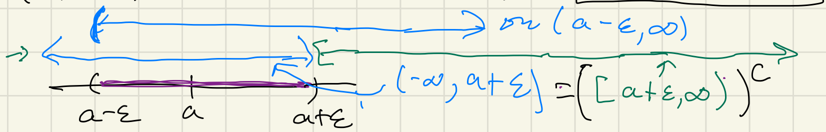
- $\mathcal{T}_{\mathbb{R}}$ is the collection of all open subsets of $\mathbb{R}$. < $\mathcal{T}$ for topology >.
- $\mathcal{B}_{\mathbb{R}} \overset{\text{or}}{=} \mathcal{B} := \sigma(\mathcal{T}_{\mathbb{R}}) :=$ Borel subsets of $\mathbb{R}$.
  ↑ if $\mathbb{R}$ is understood

Useful Facts: "ε-NBHDS & ½-rays are generating sets for $\mathcal{B}$."

(1) HW: $\mathcal{B}_{\mathbb{R}} = \sigma(\overbrace{\{N_\varepsilon(a) : \varepsilon > 0,\ a \in \mathbb{R}\}}^{:=\mathcal{E}})$. <going to assume now>

using (1), we will show

(2) Also: $\mathcal{B}_{\mathbb{R}} = \sigma(\overbrace{\{(a,\infty) : a \in \mathbb{R}\}}^{:=\mathcal{R}})$, ie $\boxed{\sigma(\mathcal{E}) = \sigma(\mathcal{R})}$



$\boxed{\sigma(\mathcal{E}) \subseteq \sigma(\mathcal{R})}$

W

$$N_\varepsilon(a) = (a-\varepsilon,\infty) \cap [a+\varepsilon,\infty)^c = \underbrace{(a-\varepsilon,\infty)}_{\mathcal{R}} \cap \underbrace{\Big[\bigcap_{n=1}^{\infty} \underbrace{(r_n(a+\varepsilon),\infty)}_{\mathcal{R}}\Big]}_{\sigma(\mathcal{R})} \in \sigma(\mathcal{R})$$

Let $0 < r_n \nearrow 1$

So $N_\varepsilon(a) \in \sigma(\mathcal{R})$

$\Downarrow$

$\mathcal{E} \subseteq \sigma(\mathcal{R})$

$\Downarrow$

$\sigma(\mathcal{E}) \subseteq \sigma(\sigma(\mathcal{R})) \overset{\text{note}}{=} \sigma(\mathcal{R})$.

$\boxed{\sigma(\mathcal{R}) \subseteq \sigma(\mathcal{E})}$.

$$(a,\infty) = \bigcup_{n=1}^{\infty} \underbrace{(a, a+n)}_{\in \mathcal{E}} \in \sigma(\mathcal{E})$$

$\Downarrow$

$\mathcal{R} \subset \sigma(\mathcal{E})$

$\Downarrow$

$\sigma(\mathcal{R}) \subseteq \sigma(\sigma(\mathcal{E})) \overset{\text{note}}{=} \sigma(\mathcal{E})$.

Rmk Proof that (1) ⇒ (2) illustrates an important techique use when dealing w/ σ-algebras.

~~2B4~~

2.35 Definition ***measurable function*** $\mathcal{S}$-msrable fn

Suppose $(X, \mathcal{S})$ is a measurable space. A function $f\colon X \to \mathbf{R}$ is called *$\mathcal{S}$-measurable* (or just *measurable* if $\mathcal{S}$ is clear from the context) if

$$f^{-1}(B) \in \mathcal{S}$$

for every Borel set $B \subset \mathbf{R}$.

Think of as for Σ-msrable fn

Have $f:(X,\Sigma) \to (\mathbb{R},\mathcal{B})$

$\qquad\qquad\quad \rotatebox{90}{$\in$} \qquad\qquad\qquad\qquad \rotatebox{90}{$\in$}$

Want $\underbrace{f^{-1}(B)}_{2^{nd}} \leftarrow \underbrace{\forall\ B}_{1^{st}}$

2.40 Definition ***Borel measurable function***

Suppose $X \subset \mathbf{R}$. A function $f\colon X \to \mathbf{R}$ is called *Borel measurable* if $f^{-1}(B)$ is a Borel set for every Borel set $B \subset \mathbf{R}$.

$\overset{\mathbb{R}}{\rotatebox{90}{$\subseteq$}}$

Have $f: X \to \mathbb{R}$.

f is Borel msrable $\iff$

$(\forall B \in \mathcal{B}_{\mathbb{R}})[f^{-1}(\mathcal{B}) \in \mathcal{B}_{\mathbb{R}}]$

2.37 Definition ***characteristic function;*** χ_E

Suppose E is a subset of a set X. The *characteristic function* of E is the function $\chi_E\colon X \to \mathbf{R}$ defined by

$$\chi_E(x) = \begin{cases} 1 & \text{if } x \in E, \\ 0 & \text{if } x \notin E. \end{cases}$$

other notation

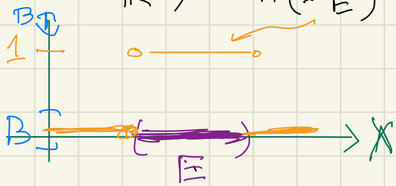
$\chi_E = \mathbb{1}_E = 1_E$.

2.38^+. Let (X, Σ) msrable space and $E \subset X$, $\langle$so $1_E : X \to \mathbb{R}\rangle$. Then 1_E is Σ-msrable $\iff E \in \Sigma$.

why? If $B \in \mathcal{B}_{\mathbb{R}}$, then $(1_E)^{-1}(B) = \{\emptyset, E, X\setminus E, X\} \overset{\text{WTH}}{\boxed{\subseteq}} \Sigma$

$\uparrow$ holds

$\boxed{\text{iff}}\ E \in \Sigma$

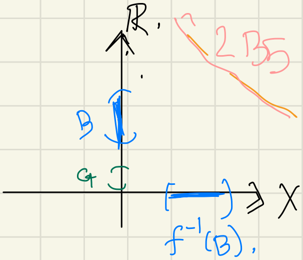


2.39 *condition for measurable function*

Suppose $(X, \mathcal{S})$ is a measurable space and $f\colon X \to \mathbf{R}$ is a function such that

$$f^{-1}((a,\infty)) \in \mathcal{S} \qquad (\mathcal{A})$$

for all $a \in \mathbf{R}$. Then f is an $\mathcal{S}$-measurable function.



One proof is in the book. We'll do another way, using the HW:

Proof of 2.39 LTGDG. Let
$$\mathcal{G} := \{G \subset \mathbb{R} \mid f^{-1}(G) \in \mathcal{S}\}.$$

Enuf to show $\mathcal{B}_{\mathbb{R}} \subseteq \mathcal{G}$.

Claim $\mathcal{G}$ is a σ-algebra

- $\emptyset \in \mathcal{G}$ b/c $f^{-1}(\emptyset) = \emptyset$ and, b/c $\mathcal{S}$ is a σ-alg, $\emptyset \in \mathcal{S}$.
- $\mathcal{G}$ is closed under complementation since, $\forall G \in \mathcal{G}$, $f^{-1}(\mathbb{R}\setminus G) = X \setminus \underbrace{f^{-1}(G)}_{\in \mathcal{S} \text{ by def of } \mathcal{G}} \underbrace{\in \mathcal{S}}_{\text{since } \mathcal{S} \text{ is closed under comp.}}$. So, by def of $\mathcal{G}$, $\mathbb{R}\setminus G \in \mathcal{G}$.
- $\mathcal{G}$ is closed under ctb. unions since, $\forall$ G_k's $\in \mathcal{G}$. $f^{-1}\left(\bigcup_{k=1}^{\infty} G_k\right) = \bigcup_{k=1}^{\infty} \underbrace{f^{-1}(G_k)}_{\in \mathcal{S} \text{ by def of } \mathcal{G} \text{ \& since } \mathcal{S} \text{ is closed under ctb. unions}} \in \mathcal{S}$. So, by def of $\mathcal{G}$, $\bigcup_{k=1}^{\infty} G_k \in \mathcal{G}$.

Let $\mathcal{R} := \{(a,\infty) \subset \mathbb{R} \mid a \in \mathbb{R}\}$.

Our assumption $(\mathcal{A})$ is $\mathcal{R} \subseteq \mathcal{G}$ $\Downarrow$ b/c $\mathcal{G}$ is a σ-alg

$$\mathcal{B}_{\mathbb{R}} \overset{\text{showed earlier}}{=} \sigma(\mathcal{R}) \subseteq \sigma(\mathcal{G}) = \mathcal{G}.$$

So $\mathcal{B}_{\mathbb{R}} \subset \mathcal{G}$. So if $B \in \mathcal{B}_{\mathbb{R}}$, then $B \in \mathcal{G}$ so, by def of $\mathcal{G}$, $f^{-1}(B) \in \mathcal{S}$. So f is $\mathcal{S}$-msr-able (by def of $\mathcal{S}$-msrable). ∎

Rmk Note this technique, commonly used w/ σ-alg.
To show: $(\forall B \in \mathcal{B})\,[f^{-1}(B) \in \mathcal{S}]$ we did NOT start by "Fix $B \in \mathcal{B}$",
Instead we looked at the "good" collection
$$\mathcal{G} := \{G \subset \mathbb{R} \mid f^{-1}(B) \in \mathcal{S}\}$$
and showed $\mathcal{G}$ is a σ-alg AND
$$(\underbrace{\text{a generating set for } \mathcal{B}}_{\text{eg, } 1/2 \text{ rays, or } \varepsilon\text{-NBHD}}) \subseteq \mathcal{S}.$$

2.41 *every continuous function is Borel measurable*

Every continuous real-valued function defined on a Borel subset of **R** is a Borel measurable function.

why? $\underset{\omega}{B_{\mathbb{R}}}$ 2B6

let $f: X \to \mathbb{R}$ cont.

WTS f is Borel msrable

well = Thinkingland.

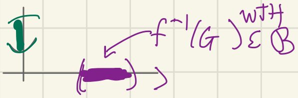
f is cont. $\iff (\forall Y_0 \text{ open in } \mathbb{R})[f^{-1}(Y_0) \text{ is open in } X]$.

$\iff (\forall N_\varepsilon(y) \subset \mathbb{R})[f^{-1}(N_\varepsilon(y)) \text{ is open in } X]$

$\iff (\forall N_\varepsilon(y) \subset \mathbb{R})(\exists O \text{ open in } \mathbb{R})[f^{-1}(N_\varepsilon(y)) = O \cap X]$.

Let $\mathcal{G} := \{G \subset \mathbb{R} \mid f^{-1}(G) \in \mathcal{B}\}_{\mathcal{G} \ni G}$

So WTS $\boxed{\mathcal{B} \subset \mathcal{G}}$.



Claim 1 $\mathcal{G}$ is σ-alg. <see pf of 2.39>

Let $\mathcal{E} := \{N_\varepsilon(y) \subset \mathbb{R} \mid \varepsilon > 0, y \in \mathbb{R}\}$. ← "range space"

Claim 2 Since $\mathcal{E}$ is a generating set for $\mathcal{B}$ <i.e. $\sigma(\mathcal{E}) = \mathcal{B}$>, Enuf TS $\boxed{\mathcal{E} \subset \mathcal{G}.}$

$B_{\mathbb{R}} = \sigma(\mathcal{E}) \subseteq \sigma(\mathcal{G}) = \mathcal{G}$ ← since $\mathcal{G}$ is a σ-alg. <claim 1>

So if $B \in B_{\mathbb{R}}$, then $B \in \mathcal{G}$ and so, by def of $\mathcal{G}$, $f^{-1}(B) \in \mathcal{B}$. ⇐ f is $\mathcal{B}$-msr.

Claim 3 Fix $N_\varepsilon(y) \in \mathcal{E}$. Then $N_\varepsilon(y) \in \mathcal{G}$.

$f: X \to \mathbb{R}$ cont $\Rightarrow f^{-1}(N_\varepsilon(y))$ is open in X

$\Rightarrow \exists\, O$ open in $\mathbb{R}$, s.t. $f^{-1}(N_\varepsilon(y)) = O \cap X \in B_{\mathbb{R}}$

so $O \in B_{\mathbb{R}}$; by assumption $X \in B_{\mathbb{R}}$

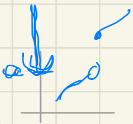
So $f^{-1}(N_\varepsilon(y)) \in B_{\mathbb{R}}$. So $N_\varepsilon(y) \in \mathcal{G}$.

∎

> **2.42 Definition** *increasing function*
>
> Suppose $X \subset \mathbf{R}$ and $f: X \to \mathbf{R}$ is a function.
>
> - f is called *increasing* if $f(x) \le f(y)$ for all $x, y \in X$ with $x < y$.
> - f is called *strictly increasing* if $f(x) < f(y)$ for all $x, y \in X$ with $x < y$.

so inc. is really "non-dec".

but nice/standard convension for sanity reasons!



2.43 *every increasing function is Borel measurable*

Every increasing function defined on a Borel subset of **R** is a Borel measurable function.

Why? Let $f: \underset{\omega}{\overset{\mathcal{B}}{X}} \xrightarrow{\text{inc.}} \mathbb{R}$.

WTS f is Borel msr.

Similarly to proof of 2.41, let $\mathcal{G} := \{G \subset \mathbb{R} : f^{-1}(G) \in \mathcal{B}\}$.

Need to show $\mathcal{B} \subset \mathcal{G}$. For a generating set for $\mathcal{B}$, instead of $\mathcal{E}$ ($= \varepsilon$-NBHD) here take rays $\mathcal{R} := \{(a, \infty) : a \in \mathbb{R}\}$. Since $f \uparrow$ $\underset{\omega}{\mathcal{B}}$, $f^{-1}((a,\infty))$ is of the form $(\underbrace{\inf f^{-1}(a,\infty)}_{\in \mathcal{B}}, \infty) \cap X$ or $[\inf f^{-1}(a,\infty), \infty) \cap X$.

$\overset{m}{\mathcal{B}}$ $\qquad \overset{\shortparallel}{\bigcap_{n=1}^{\infty} ((\inf f^{-1}(a,\infty)) - \tfrac{1}{n}, \infty)} \in \mathcal{B}$

So $f^{-1}((a,\infty)) \in \mathcal{B}$.

So $(a,\infty) \in \mathcal{G}$.

2 B7

2.44 *composition of measurable functions*

Suppose $(X, \mathcal{S})$ is a measurable space and $f \colon X \to \mathbf{R}$ is an $\mathcal{S}$-measurable function. Suppose g is a real-valued Borel measurable function defined on a subset of $\mathbf{R}$ that includes the range of f. Then $g \circ f \colon X \to \mathbf{R}$ is an $\mathcal{S}$-measurable function.

$$\begin{tikzcd}
\text{Let } f:(X,\mathcal{S}) \arrow[r, "\mathcal{S}\text{-msr}"] & \mathbb{R} & \\
 & \cup\!| & \\
 & f[X] & \\
 & \cap\!| & \\
 & g: D \arrow[r, "\text{Borel-msr}"] & \mathbb{R}. \\
 & \cap\!| & \\
 & \mathbb{R} &
\end{tikzcd}$$

$$\Rightarrow \boxed{g\circ f : (X,\mathcal{S}) \xrightarrow{f} \overset{\overset{\mathbb{R}}{\cup|}}{f[X]} \xrightarrow{g} \mathbb{R}}$$

is well-defined by

$$(g\circ f)(x) := g(f(x))$$

WTS $g\circ f$ is $\mathcal{S}$-msr. Fix $B \in \mathcal{B}$. Enuf to show $(g\circ f)^{-1}(B) \in \mathcal{S}$.

$$g\circ f : (X, \mathcal{S}) \xrightarrow{f} f[X] \xrightarrow{g} \mathbb{R}$$

$$\begin{tikzcd}
(g\circ f)^{-1}(B) & f^{-1}(g^{-1}(B)) \arrow[l, Leftarrow, "④", "\text{note}"'] & g^{-1}(B) \arrow[l, Leftarrow, "③", "f\ \mathcal{S}\text{-msr}"'] & \overset{①}{B}. \arrow[l, Leftarrow, "②", "g\ \mathcal{B}\text{-msr}"'] \\
 & \in \mathcal{S} & \in \mathcal{B} & \in \mathcal{B}
\end{tikzcd}$$

∎

2.45 Cor. Let $f:(X,\mathcal{S}) \to \mathbb{R}$ be $\mathcal{S}$-msr. Let $c \in \mathbb{R}$ & $n \in \mathbb{N}$. Then

1. cf is $\mathcal{S}$-msr
2. $|f|$ is $\mathcal{S}$-msr
3. f^n is $\mathcal{S}$-msr.

Why? In 2.44. take $g: \mathbb{R} \to \mathbb{R}$ to be:

$$\left.\begin{aligned} g(x) &= cx \\ g(x) &= |x| \\ g(x) &= x^n. \end{aligned}\right\}$$

Recall, cont. fns. $g: \mathbb{R} \to \mathbb{R}$ are Borel msr. (2.41)

and if futhermore $0 \notin f[X]$,

4. $\frac{1}{f}$ is $\mathcal{S}$-msr

Take $g: \mathbb{R}\setminus\{0\} \to \mathbb{R}$ to be $g(x) = \frac{1}{x}$.

$\boxed{\text{L2B8}}$

Work Horse If $f, g : \underbrace{(X, \mathcal{S})}_{\text{msr-able space.}} \to \mathbb{R}$ are $\mathcal{S}$-msr, then $f+g$ is $\mathcal{S}$-msr.

Why?

Need to show $\forall\, B \in \mathcal{B}$, $(f+g)^{-1}(B) \in \mathcal{S}$.

Since $\mathcal{B} = \sigma(\{(a,\infty) : a \in \mathbb{R}\})$, EnufToShow $\forall\, a \in \mathbb{R}$, $(f+g)^{-1}((a,\infty)) \in \mathcal{S}$

Fix $a \in \mathbb{R}$.

$$\begin{aligned}
x \in (f+g)^{-1}((a,\infty)) &\iff (f+g)(x) \in (a,\infty) \\
&\iff a < f(x) + g(x) \\
&\iff a - g(x) < f(x) \\
&\iff \exists\, r \in \mathbb{Q} \text{ s.t. } a - g(x) < r \quad \text{and} \quad r < f(x) \\
&\iff \quad '' \quad a - r < g(x) \quad \text{and} \quad r < f(x) \\
&\iff \quad '' \quad g(x) \in (a-r,\infty) \quad \text{and} \quad f(x) \in (r,\infty) \\
&\iff \quad '' \quad x \in \left[g^{-1}((a-r,\infty)) \cap f^{-1}((r,\infty)) \right]
\end{aligned}$$

So

$$(f+g)^{-1}((a,\infty)) = \bigcup_{r \in \mathbb{Q}} \Big[\underbrace{g^{-1}((a-r,\infty))}_{\in\, \mathcal{S} \text{ b/c } g \text{ is } \mathcal{S}\text{-msr}} \cap \underbrace{f^{-1}((r,\infty))}_{\in\, \mathcal{S} \text{ bk } f \text{ is } \mathcal{S}\text{-msr}} \Big] \underset{\Uparrow}{\in} \mathcal{S}.$$

countable ↑

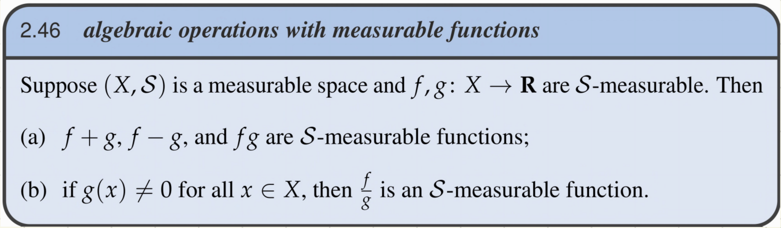
~~2B9~~

Recall:

2.45 Cor. Let $f:(X,\mathcal{S})\to\mathbb{R}$ be $\mathcal{S}$-msr. Let $c\in\mathbb{R}$ & $n\in\mathbb{N}$. Then

1. cf is $\mathcal{S}$-msr
2. $|f|$ is $\mathcal{S}$-msr
3. f^n is $\mathcal{S}$-msr.
4. $\frac{1}{f}$ is $\mathcal{S}$-msr, provided $0\notin f[X]$.

(WH) Work Horse If $f,g:\underbrace{(X,\mathcal{S})}_{\text{msr-able space.}}\to\mathbb{R}$ are $\mathcal{S}$-msr, then $f+g$ is $\mathcal{S}$-msrable



2.46 algebraic operations with measurable functions

Suppose $(X,\mathcal{S})$ is a measurable space and $f,g\colon X\to\mathbf{R}$ are $\mathcal{S}$-measurable. Then

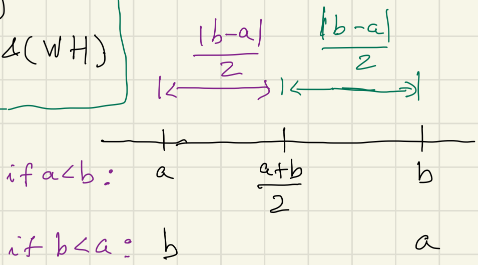
(a) $f+g$, $f-g$, and fg are $\mathcal{S}$-measurable functions;

(b) if $g(x)\neq 0$ for all $x\in X$, then $\frac{f}{g}$ is an $\mathcal{S}$-measurable function.

(c) max(f,g) and min(f,g) are $\mathcal{S}$-msr-able

- $f+g$ is the Work Horse (WH)
- $f-g = f+(-1)g$ so use (1) & (WH)
- $fg = \dfrac{(f+g)^2-f^2-g^2}{2}$.
- $\dfrac{f}{g} = (f)\left(\dfrac{1}{g}\right)$
- For $a,b\in\mathbb{R}$:

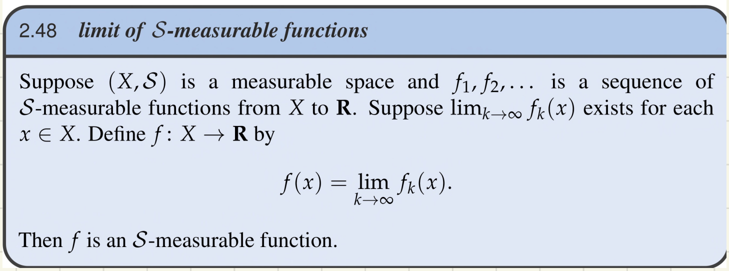
For $a,b\in\mathbb{R}$



also works if a=b.

$$\max(a,b)=\frac{a+b}{2}+\frac{|b-a|}{2} \quad \text{and} \quad \min(a,b)=\frac{a+b}{2}-\frac{|b-a|}{2}$$

So, eg, $(\max(f,g))(x) = \frac{1}{2}(f+g)(x) - \frac{1}{2}|f-g|(x)$.



2.48 limit of $\mathcal{S}$-measurable functions

Suppose $(X,\mathcal{S})$ is a measurable space and $f_1,f_2,\dots$ is a sequence of $\mathcal{S}$-measurable functions from X to $\mathbf{R}$. Suppose $\lim_{k\to\infty}f_k(x)$ exists for each $x\in X$. Define $f\colon X\to\mathbf{R}$ by

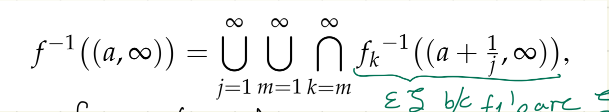
$$f(x)=\lim_{k\to\infty}f_k(x).$$

Then f is an $\mathcal{S}$-measurable function.

Enuf to show

$$f^{-1}(\underbrace{\text{"rays"}}_{\text{generating set}})\in\mathcal{S}.$$

Why? For $a\in\mathbb{R}$,



$$f^{-1}((a,\infty))=\bigcup_{j=1}^{\infty}\bigcup_{m=1}^{\infty}\bigcap_{k=m}^{\infty}\underbrace{f_k^{-1}((a+\tfrac{1}{j},\infty))}_{\in\mathcal{S}\text{ b/c }f_k\text{'s are }\mathcal{S}\text{-msr}},\in\mathcal{S}.$$

Note $x\in f^{-1}((a,\infty)) \iff f(x)\in(a,\infty)$

$\iff a<\lim f_k(x)$

$\iff (\exists j\in\mathbb{N})\,[a+\frac{1}{j}<\lim f_k(x)]$

$\iff (\exists j\in\mathbb{N})(\exists m\in\mathbb{N})(\forall k\in\mathbb{N}^{\geq m})\,[a+\frac{1}{j}<f_k(x)]$

2B 10

2.50 **Definition** *Borel subsets of* $[-\infty,\infty]$ $= \mathcal{B}_{\hat{\mathbb{R}}}$.

A subset of $[-\infty,\infty]$ is called a *Borel set* if its intersection with $\mathbf{R}$ is a Borel set.

$\hat{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\} = [-\infty,\infty]$

i.e. $\mathcal{B}_{\hat{\mathbb{R}}} := \mathcal{B}_{\mathbb{R}} \cup \{B \cup \{\infty\} : B \in \mathcal{B}_{\mathbb{R}}\} \cup \{B \cup \{-\infty\} : B \in \mathcal{B}_{\mathbb{R}}\} \cup \{B \cup \{\infty,-\infty\} : B \in \mathcal{B}_{\mathbb{R}}\}$.

For You Verify $\mathcal{B}_{\hat{\mathbb{R}}}$ is a σ-algebra.

2.51 **Definition** *measurable function*

Suppose $(X,\mathcal{S})$ is a measurable space. A function $f\colon X \to [-\infty,\infty]$ is called $\mathcal{S}$*-measurable* if

$$f^{-1}(B) \in \mathcal{S}$$

for every Borel set $B \subset [-\infty,\infty]$., i.e. $\forall B \in \mathcal{B}_{\hat{\mathbb{R}}}$

- For You Let $f : (X, \mathcal{S}) \to \hat{\mathbb{R}}$ w/ $(X, \mathcal{S})$ a msrable space. Then.

$$f \text{ is } \mathcal{S}\text{-msrable} \iff \left[\begin{array}{l} \bullet\ f|_{f^{-1}(\mathbb{R})} : f^{-1}(\mathbb{R}) \to \mathbb{R} \text{ is } \mathcal{S}\text{-msr} \\ \bullet\ f^{-1}(\{\infty\}) \in \mathcal{S} \\ \bullet\ f^{-1}(\{-\infty\}) \in \mathcal{S}. \end{array}\right]$$

- 2.52 says: to check msr-ablity, it suffices to check sets in a generating collection.

2.52 *condition for measurable function*

Suppose $(X,\mathcal{S})$ is a measurable space and $f\colon X \to [-\infty,\infty]$ is a function such that

$$f^{-1}((a,\underline{\infty}]) \in \mathcal{S}$$

for all $a \in \mathbf{R}$. Then f is an $\mathcal{S}$-measurable function.

2.53* *infimum and supremum of a sequence of* $\mathcal{S}$*-measurable functions*

Suppose $(X,\mathcal{S})$ is a measurable space and $f_1, f_2, \ldots$ is a sequence of $\mathcal{S}$-measurable functions from X to $[-\infty,\infty]$. Define $g, h\colon X \to [-\infty,\infty]$ by

$$g(x) = \inf\{f_k(x) : k \in \mathbf{Z}^+\} \quad \text{and} \quad h(x) = \sup\{f_k(x) : k \in \mathbf{Z}^+\}.$$

Then g and h are $\mathcal{S}$-measurable functions. Also $\overline{\lim_{n\to\infty}} f_n$ & $\underline{\lim_n} f_n$ are $\mathcal{S}$-msr.

why? $\forall a \in \mathbb{R}$:

$$h^{-1}((a,\infty]) = \bigcup_{k=1}^{\infty} {f_k}^{-1}((a,\infty])$$

$$g(x) = -\sup\{-f_k(x) : k \in \mathbf{Z}^+\}$$

Futhermore, if $\lim_{n\to\infty} f : X \to \hat{\mathbb{R}}$ exists, then $\lim_{n\to\infty} f_n$ is $\mathcal{S}$-msr.:

- $$\overline{\lim_{n\to\infty}} f_n(x) := \lim_{k\to\infty} \underbrace{\left[\sup_{n \in \mathbb{N}^{\geq k}} f_n(x)\right]}_{\text{as } k \nearrow,\ \text{this } \searrow} = \inf_{k \in \mathbb{N}} \left[\sup_{n \in \mathbb{N}^{\geq k}} f_n(x)\right].$$

- $\underline{\lim} f_n = -\overline{\lim} -f_n$.
- If $\lim_{n\to\infty} f_n : X \to \hat{\mathbb{R}}$ exists, then $\overline{\lim} f_n = \underline{\lim} f_n = \lim f_n$.