Ch 2 Measure

§ 2A Outer Measure on $\mathbb{R}$

2.1 Definition ***length of open interval;*** $\ell(I)$

The *length* $\ell(I)$ of an open interval I is defined by

$$\ell(I) = \begin{cases} b-a & \text{if } I=(a,b) \text{ for some } a,b\in\mathbf{R} \text{ with } a<b, \\ 0 & \text{if } I=\varnothing, \\ \infty & \text{if } I=(-\infty,a) \text{ or } I=(a,\infty) \text{ for some } a\in\mathbf{R}, \\ \infty & \text{if } I=(-\infty,\infty). \end{cases}$$

2.2 Definition ***outer measure;*** $|A|$

The *outer measure* $|A|$ of a set $A\subset\mathbf{R}$ is defined by

$$|A| = \inf\Big\{\sum_{k=1}^{\infty}\ell(I_k) : I_1, I_2, \dots \text{ are open intervals such that } A\subset\bigcup_{k=1}^{\infty} I_k\Big\}.$$

Note: $I_k = \emptyset$ is allowed.

Good Properties of outer measure

2.4 ***countable sets have outer measure*** 0

Every countable subset of $\mathbf{R}$ has outer measure 0.

Why?

2.5 *outer measure preserves order*

Suppose A and B are subsets of $\mathbf{R}$ with $A \subset B$. Then $|A| \leq |B|$.

Why?

Let $\mathcal{A} = \{ \{I_i\}_{i \in \mathbb{N}} \mid I_i \text{ are open intervals and } A \subset \bigcup_i I_i \}$

$\mathcal{B} = \{ \{I_i\}_{i \in \mathbb{N}} \mid I_i \text{ are open intervals and } B \subset \bigcup_i I_i \}$.

Note $A \subset B \Rightarrow \mathcal{A} \boxed{} \mathcal{B}$

Now

$$|A| := \inf_{\{I_i\} \in \mathcal{A}} \sum_i \ell(I_i) \;\boxed{}\; \inf_{\{I_i\} \in \mathcal{B}} \sum_i \ell(I_i) =: |B|$$

2.6 Definition ***translation;*** $t + A$

If $t \in \mathbf{R}$ and $A \subset \mathbf{R}$, then the *translation* $t + A$ is defined by

$$t + A = \{t + a : a \in A\}.$$

So

$$t + A = \{t\} \overset{\text{Minkowski sum}}{+} A$$

2.7 *outer measure is translation invariant*

Suppose $t \in \mathbf{R}$ and $A \subset \mathbf{R}$. Then $|t + A| = |A|$.

Why?

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2.8 *countable subadditivity of outer measure*

Suppose $A_1, A_2, \ldots$ is a sequence of subsets of $\mathbf{R}$. Then

$$\left|\bigcup_{k=1}^{\infty} A_k\right| \leq \sum_{k=1}^{\infty} |A_k|.$$

2.14 ***outer measure of a closed interval*** $|[a,b]| = \ell([a,b])$

Suppose $a, b \in \mathbf{R}$, with $a < b$. Then $|[a,b]| = b - a$.

2.17 ***nontrivial intervals are uncountable***

Every interval in $\mathbf{R}$ that contains at least two distinct elements is uncountable.

See book's (short) proof, which uses measures.

Not Good Property of outer measure

Unforunately, outer measure is not additive.

2.18 *nonadditivity of outer measure*

There exist disjoint subsets A and B of $\mathbf{R}$ such that

$$|A \cup B| \neq |A| + |B|.$$

In fact.

2.22 *nonexistence of extension of length to all subsets of* R

There does not exist a function μ with all the following properties:

(a) μ is a function from the set of subsets of $\mathbf{R}$ to $[0, \infty]$.

(b) $\mu(I) = \ell(I)$ for every open interval I of $\mathbf{R}$.

(c) $\mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \mu(A_k)$ for every disjoint sequence $A_1, A_2, \ldots$ of subsets of $\mathbf{R}$.

(d) $\mu(t + A) = \mu(A)$ for every $A \subset \mathbf{R}$ and every $t \in \mathbf{R}$.

property (c) is called countably additive.

So here our game plan