Observation 1. Let $z \in \mathbb{C}$. TFAE, as easily seen by writing $z = x + iy \in \mathbb{C}$ where $x = \operatorname{Re} z \in \mathbb{R}$ and $y = \operatorname{Im} z \in \mathbb{R}$.

(1.1) $z \geq 0$ (a convenient way to indicate that z is real and nonnegative, i.e. $z = \operatorname{Re} z \geq 0$)
(1.2) $z = |z|$
(1.3) $\operatorname{Re} z = |z|$

Observation 2. Let $z_1, z_2 \in \mathbb{C}$. TFAE. (think what this is saying geometrically)

(2.1) $z_1 \overline{z_2} \geq 0$
(2.2) $z_1 \, \overline{z_2} = |z_1| \, |z_2|$
(2.3) $\operatorname{Re} z_1 \overline{z_2} = |z_1 \overline{z_2}|$
(2.4) $[\ z_2 = 0\ ]$ or $\left[\ \frac{z_1}{z_2} \geq 0\ \right]$
(2.5) $[\ z_2 = 0\ ]$ or $[\ z_1 = \lambda z_2$ for some $\lambda \in [0, \infty)\ ]$

Theorem 1. [Triangle Inequality (with equality)] Let $n \in \mathbb{N}$ and $z_1, \ldots, z_n$ from $\mathbb{C}$. Then

$$|z_1 + \ldots + z_n| \ \leq \ |z_1| + \ldots + |z_n| \tag{1}$$

and equality holds in (1) if and only if

$$z_j \, \overline{z_k} \ = \ |z_j| \, |z_k| \qquad \text{for each} \qquad j, k \in \mathbb{N}^{\leq n} \text{ with } j \neq k. \tag{2}$$

An equivalent formuation of (2) is

$$z_j \, \overline{z_k} \ = \ |z_j| \, |z_k| \qquad \text{for each} \quad j, k \in \mathbb{N}^{\leq n}. \tag{2'}$$

Proof. Let $n \in \mathbb{N}$. The equivalence of (2) and (2′) follows from $z\overline{z} = |z|^2$ for any $z \in \mathbb{C}$. Theorem 1 clearly holds when $n = 1$ for any $z_1 \in \mathbb{C}$. Thus we assume $n \geq 1$.

First we show inequality (1) by induction. Let $n = 2$. Fix any $z_1, z_2 \in \mathbb{C}$. Then (1) holds aince

$$\begin{aligned} |z_1 + z_2|^2 &= (z_1 + z_2)\, \overline{(z_1 + z_2)} \\ &= (z_1 + z_2)\, (\overline{z_1} + \overline{z_2}) \\ &= z_1\, \overline{z_1} + z_2\, \overline{z_2} + z_1\, \overline{z_2} + \overline{z_1}\, z_2 \\ &= |z_1|^2 + |z_2|^2 + z_1\, \overline{z_2} + \overline{z_1\, \overline{z_2}} \\ &= |z_1|^2 + |z_2|^2 + 2\text{Re}\, (z_1\, \overline{z_2}) \\ &\overset{(*)}{\leq} |z_1|^2 + |z_2|^2 + 2\, |z_1\, \overline{z_2}| \\ &= |z_1|^2 + |z_2|^2 + 2\, |z_1|\, |z_2| \\ &= (|z_1| + |z_2|)^2 . \end{aligned} \tag{3}$$

Now fix $n \in \mathbb{N}^{\geq 2}$. Assume that Theorem 1 holds for any collection of $m \in \mathbb{N}$ complex numbers where $m \leq n$. Fix $z_1, \ldots, z_{n+1}$ from $\mathbb{C}$. Then

$$\left| \sum_{j=1}^{n+1} z_j \right| \ = \ \left| \left(\sum_{j=1}^{n} z_j \right) + z_{n+1} \right| \ \leq \ \left| \sum_{j=1}^{n} z_j \right| + |z_{n+1}| \ \leq \ \left(\sum_{j=1}^{n} |z_j| \right) + |z_{n+1}| \ = \ \sum_{j=1}^{n=1} |z_n| \, .$$

Thus (1) holds.

Thus, for any $n \in \mathbb{N}$ and $z_1, \ldots, z_n$ from $\mathbb{C}$, inequality (1) holds.

The remaninder of the proof is the next Exercise. $\square$

Outline of rest of proof ...

What remains to be shown is, $\forall n \in \mathbb{N}$,

$$\left| \sum_{i=1}^{n} z_i \right| = \sum_{i=1}^{n} |z_i| \iff (2) \text{ holds.} \qquad (\text{WTS})$$

- If $n=1$, then (WTS) clearly holds for any $z_1 \in \mathbb{C}$.
- Let $n=2$. Note

$$|z_1 + z_2| = |z_1| + |z_2|$$

is equiv. to, by the calculation in (3) $\langle$namely the $\overset{(*)}{\leq}$$\rangle$

$$\operatorname{Re}(z_1 \overline{z}_2) = |z_1 \overline{z}_2|$$

which is equiv to, by Observation 2,

$$z_1 \overline{z}_2 = |z_1|\,|z_2| .$$

So (WTS) holds for $n=2$ for any $z_1, z_2 \in \mathbb{C}$.

- $\boxed{\text{WTS} \Rightarrow \text{for } n \geq 3}$ Fix $n \in \mathbb{N}^{\geq 3}$. Let

$$\left| \sum_{i=1}^{n} z_i \right| = \sum_{i=1}^{n} |z_i| . \qquad (4)$$

$\langle$WTS (2) holds.$\rangle$

Fix $j, k \in \mathbb{N}^{\leq n}$ such that $j \neq k$. Let

$$\Gamma = \{ i \in \mathbb{N}^{\leq n} : i \neq j \text{ and } i \neq k \} .$$

$\langle$Note card $\Gamma \geq 1$ and $\Gamma \uplus \{j\} \uplus \{k\} = \{1, 2, \ldots, n\}$$\rangle$

So

$$|z_j|+|z_k|+\sum_{i\in\Gamma}|z_i| = \sum_{i=1}^{n}|z_i|$$

using our assumption.

$$\overset{\text{by }(4)}{=} \left|\sum_{i=1}^{n} z_i\right|$$

$$= \left|(z_j+z_k)+\sum_{i\in\Gamma} z_i\right|$$

$$\underset{(1)}{\overset{\Delta}{\leq}} |z_j+z_k|+\sum_{i\in\Gamma}|z_i| .$$

So $|z_j|+|z_k| = |z_j+z_k|$. Now apply $n=2$ case of (WTS) to get (2) holds for j & k (which were arbitrary so done).

$\boxed{\text{WTS} \Leftarrow \text{ for } n\geq 3}$ Fix $n\in\mathbb{N}^{\geq 3}$. Let

$$z_j\overline{z_k} = |z_j|\,|z_k| , \quad \forall j,k\in\mathbb{N}^{\leq n} . \qquad (2')$$

We need to show $\left|\sum_{i=1}^{n} z_i\right| = \sum_{i=1}^{n}|z_i|$ (5)

$\underline{\text{Case 1}}$ $z_j=0 \ \forall j\in\mathbb{N}^{\leq n}$. Clearly (5) holds.

$\underline{\text{Case 2}}$ $\exists k\in\mathbb{N}^{\leq n}$ st $z_k\neq 0$. So $\overline{z_k}\neq 0$. So

$$\left|\sum_{i=1}^{n} z_i\right| = \left|\frac{1}{\overline{z_k}}\sum_{i=1}^{n}(z_i\overline{z_k})\right| \overset{(2')}{=} \frac{1}{|\overline{z_k}|}\left|\sum_{i=1}^{n}|z_i|\,|\overline{z_k}|\right|$$

our assumption.

$$= \sum_{i=1}^{n}|z_i| .$$

So (5) holds. ∎

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You are **strongly** encouraged to work in groups, following the procedure as in homework MS09.

Exercise pCA 5. Variant of 3.1.24.3 (p. 173).

Read section 3.1. Then finish the proof of Theorem 1 from the previous page.