Defining the Exponential Function. Using the real exponential function $e^{(\cdot)} \colon \mathbb{R} \to \mathbb{R}$, the complex exponential function $\exp(\cdot) : \mathbb{C} \to \mathbb{C}$ is defined by

$$\exp(x + iy) := e^x \cos y + ie^x \sin y \overset{\text{i.e.}}{=} e^x (\cos y + i \sin y) \quad , \ x, y \in \mathbb{R}. \tag{1}$$

The complex exponential function restricted to $\mathbb{R}$ agrees the the usual real exponential fnc. since

$$\exp(x) = \exp(x + i0) = e^x \cos 0 + ie^x \sin 0 = e^x \quad , \ x \in \mathbb{R}. \tag{2}$$

Thus we often write $\exp(x + iy)$ by e^{x+iy}.

Defining Trigomonetric Functions. From (1) it follows that if $x \in \mathbb{R}$ then

$$e^{ix} + e^{-ix} = 2\cos x \quad \text{and} \quad e^{ix} - e^{-ix} = 2i \sin x \quad , \ z \in \mathbb{R}. \tag{3}$$

Motivated by (3), the (complex) sine and cosine functions (from $\mathbb{C}$ to $\mathbb{C}$) are defined by

$$\cos z := \frac{e^{iz} + e^{-iz}}{2} \quad \text{and} \quad \sin z := \frac{e^{iz} - e^{-iz}}{2i} \quad , \ z \in \mathbb{C}. \tag{4}$$

Defining Hyperbolic Functions. Motivated by their definition for a real number, the (complex) hyperbolic sine and cosine functions (from $\mathbb{C}$ to $\mathbb{C}$) are defined by

$$\cosh z := \frac{e^{z} + e^{-z}}{2} \quad \text{and} \quad \sinh z := \frac{e^{z} - e^{-z}}{2} \quad , \ z \in \mathbb{C}. \tag{5}$$

Good Reference. [BC] Brown and Churchill, *Complex Variables and Applications.* (any edition). From Chapter 3, read sections:

§23. The Exponential Function (p. 65-68)
§24. Trigonometric Functions (p. 69-72)
§25. Hyperolic Functions (p. 72-75).

Concentrate on the definitions and properities of these functions, ignoring for now the parts about derivatives/entire. The above reference shows that these functions are defined on the whole of $\mathbb{C}$ in such a way that

(1) when a function's domain is restricted from $\mathbb{C}$ to $\mathbb{R}$, the resulting function agrees with the function (of the same name) from $\mathbb{R}$ to $\mathbb{R}$ that we know from calculus
(2) many identities/properties which we know from the $\mathbb{R}$-version extend to the $\mathbb{C}$-version (e.g., $e^{z_1} e^{z_2} = e^{z_1 + z_2}$ for each $z_z, z_2 \in \mathbb{C}$).

You should have a working knowledge of the properties/identities of the complex exponential, sine, and cosine functions. One major difference between the complex and real exponential functions is that the complex exponential function is periodic with a pure imaginary period of $2\pi i$, i.e.,

$$\exp(z + 2\pi i) = \exp(z), \quad , \ z \in \mathbb{C}. \tag{6}$$

Thus we will have to take care when defining a complex version of an "'inverse" of the complex exponential (i.e., the complex log).

Exercise pCA 4. Find all the solutions of the equation $\sin z = 3$, expressing your solution(s) in the form $a + ib$ with $a, b \in \mathbb{R}$.

Remark. The first solution uses trig. functions. The second solution uses hyperbolic trig. functions as well as the relations between complex trig and hyperbolic trig functions.
Recall. Fix $w \in \mathbb{C} \setminus \{0\}$. Recall from class that

$$\{z \in \mathbb{C} \colon e^z = w\} \;=\; \{\; \ln|w| + i\theta \colon \theta \in \arg w \;\} \;. \qquad \text{(important)}$$

If $z \in \mathbb{C} \setminus \{z \in \mathbb{R} \colon z \leq 0\}$, then

$$[\; e^z = w \;] \qquad \Longleftrightarrow \qquad [\; z \;=\; \ln|w| + i\,(\operatorname{Arg} w + 2\pi k) \quad , \; k \in \mathbb{Z} \;] \;.$$

First Solution (using trig. functions). Let $z = x + iy$, with $x, y \in \mathbb{R}$. By definition,

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

Thus, the following are equivalent.

$$\sin(z) = 3$$
$$e^{iz} - e^{-iz} = 6i$$
$$e^{iz} - 6i - e^{-iz} = 0$$
$$\left(e^{iz}\right)^2 - 6i\left(e^{iz}\right) - 1 = 0 \;.$$

Note that

$$(-6i)^2 - 4(1)(-1) \;=\; -32 \;\overset{k\in\mathbb{Z}}{=}\; 4^2\, 2\, e^{i(\pi+2\pi k)} \;, \qquad (4.3)$$

and taking $k = 0$ in (4.3) gives a complex square root of -32 is

$$4\sqrt{2}\; e^{i\left(\frac{\pi}{2}\right)} \overset{\text{i.e.}}{=} 4\sqrt{2}\, i \;.$$

Note that

$$\frac{6i \pm 4\sqrt{2}\, i}{2} = \left(3 \pm 2\sqrt{2}\right) i \;.$$

Thus

$$\sin z = 3 \qquad \text{if only only if} \qquad e^{iz} = (3 \pm 2\sqrt{2})i \;.$$

So by (important), TFAE.

$$e^{iz} \;=\; (3 \pm 2\sqrt{2})i$$
$$iz \;=\; \ln\left|(3 \pm 2\sqrt{2})i\right| \;+\; i\left(\operatorname{Arg}\left(\left(3 \pm 2\sqrt{2}\right) i\right) + 2\pi k\right) \qquad , \; k \in \mathbb{Z}$$
$$iz \;=\; \ln(3 \pm 2\sqrt{2}) \;+\; i\left(\frac{\pi}{2} + 2\pi k\right) \qquad , \; k \in \mathbb{Z}$$
$$z \;=\; -i\; \ln(3 \pm 2\sqrt{2}) \;+\; \left(\frac{\pi}{2} + 2\pi k\right) \qquad , \; k \in \mathbb{Z} \;.$$

So

$$\boxed{\left\{\; \left(\tfrac{\pi}{2} + 2\pi k\right) \;-\; i\ln(3 \pm 2\sqrt{2}) \colon k \in \mathbb{Z} \;\right\}}$$

is the solution set of the equation $\sin z = 3$. ☐

Second Solution (using hyperbolic trig. functions). Let $z = x + iy$, with $x, y \in \mathbb{R}$. Consider the equation

$$\sin(z) = 3 \; . \tag{4.0}$$

Since $\sin(x+iy) = \sin(x)\cosh(y) + i\cos(x)\sinh(y)$, equation (4.0) is equivalent to the following 2 equations both holding.

$$\sin(x)\cosh(y) \;=\; 3 \tag{4.1}$$

$$\cos(x)\sinh(y) = 0 \; . \tag{4.2}$$

To see that $\sinh y$ cannot equal 0, assume $\sinh y = 0$. Then $y = 0$ and so $\cosh y = \cosh 0 = 1$. So (4.1) implies that $\sin x = 3$. But $x \in \mathbb{R}$, a $\boxed{\lightning}$. So $\sinh y \neq 0$. So z satisfies (4.2) if and only if $\cos x = 0$, equivalently,

$$x \in \left\{ \tfrac{(2k+1)\pi}{2} \colon k \in \mathbb{Z} \right\} \overset{\text{i.e.}}{=} \left\{ \tfrac{\pi}{2} + k\pi \colon k \in \mathbb{Z} \right\} \; .$$

Now we need to also satisfy (4.1). Consider $x = \frac{\pi}{2} + k\pi$ where $k \in \mathbb{Z}$. Note that that

$$\sin(x) = \begin{cases} 1 & \text{when } k \text{ is even} \\ -1 & \text{when } k \text{ is odd.} \end{cases}$$

To see that k cannot be odd, suppose that k is odd. Then (4.1) would imply that $\cosh y = -3$. But $\cosh\theta \geq 1$ for any $\theta \in \mathbb{R}$. A $\boxed{\lightning}$. So k must be even. So (4.1) is satisfies if and only if $\cosh(y) = 3$. Note that the following are equivalent.

$$\begin{aligned} \cosh(y) &= 3 \\ \tfrac{e^y + e^{-y}}{2} &= 3 \\ e^y - 6 + e^{-y} &= 0 \\ (e^y)^2 - 6e^y + 1 &= 0 \\ e^y &= \frac{6 \pm \sqrt{36-4}}{2} \\ y &= \ln\left(3 \pm 2\sqrt{2}\right) \end{aligned}$$

Thus

$$\boxed{\left\{ \left(\tfrac{\pi}{2} + k\pi\right) + i\,\ln\left(3 \pm 2\sqrt{2}\right) \,:\, k \in 2\mathbb{Z} \right\}} \quad \overset{\text{i.e}}{=} \quad \left\{ \left(\tfrac{(4k+1)\pi}{2}\right) \pm i \operatorname{arccosh} 3 \colon k \in \mathbb{Z} \right\}$$

is the solution set of (4.0). □

Remark. Note that the First Solution and Second Solution are indeed the same since

$$-\ln(3 - 2\sqrt{2}) \;=\; \ln(3 - 2\sqrt{2})^{-1} \;=\; \ln(3 + 2\sqrt{2}) \; .$$