Exercise pCA 3. Describe (in words or/and a picture) the sets whose points satisfy the following relations. Which of these sets are regions? By definition, a <u>region</u> is an open connected set; you can argue openness and connectedness intuitively (so no ε's needed).

ER 3.a. $|z+i| \leq 1$.

ER 3.b. $\left|\dfrac{z-1}{z+1}\right| = 1$.

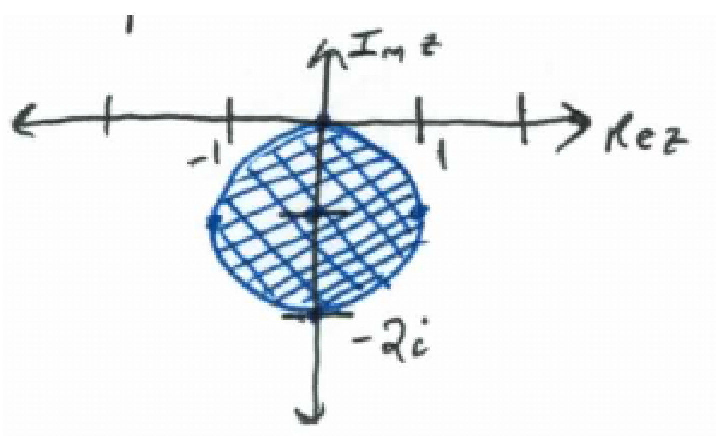
ER 3.c. $|z-3| > |z-2|$.

ER 3.d. $\dfrac{1}{z} = \overline{z}$.

$\triangleright$ Throughtout, let $z = x+iy$ or $z = re^{i\theta}$ where $x, y, r, \theta \in \mathbb{R}$ with $r > 0$.

(a). This set is the set of points $z \in \mathbb{C}$ such that the distance between z and $-i$ is ≤ 1. So this is the closed (boundary and interior points included) ball with radius 1 and center $-i$. This set is connected but is not open; thus, this set is not a region. Geometrically:

$z \in \mathbb{C}$ satisfying $|z+i| \leq 1$ is equivalent to all the $x, y \in \mathbb{R}$ satisfying $x^2 + (y+1)^2 \leq 1$, which is a closed disk in the plane with radius 1 and center at $(0,-1)$.



(b). This set is the set of points $z \in \mathbb{C} \setminus \{-1\}$ such that the distance between z and 1 is equal to the distance between z and -1. Thus this set is the imaginary axis

$$\{z \in \mathbb{C} \colon \operatorname{Re} z = 0\} \ .$$

Indeed, if $z = x + iy$ with $x, y \in \mathbb{R}$, then the following are equivalent.

$$|z-1| = |z+1|$$

$$|z-1|^2 = |z+1|^2$$

$$(x-1)^2 + y^2 = (x+1)^2 + y^2$$

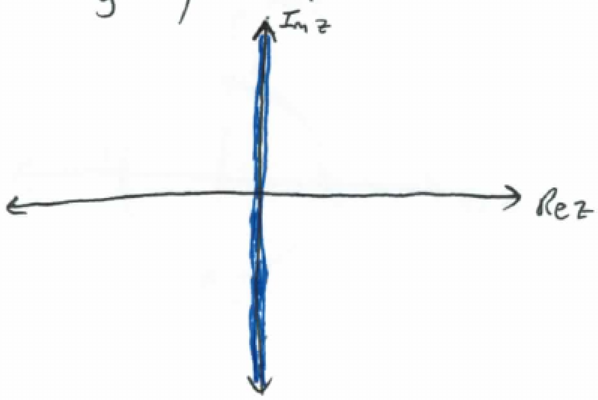
$$(x-1)^2 = (x+1)^2$$

$$x^2 - 2x + 1 = x^2 + 2x + 1$$

$$x = 0 \ .$$

This set is connected but is not open; thus, this set is not a region. Geometrically:

$z \in \mathbb{C}$ satisfying $\left|\frac{z-1}{z+1}\right| = 1$ must satisfy $|z-1| = |z+1|$, i.e. $(x-1)^2 = (x+1)^2$. Thus, if $x = 0$ and $y \in \mathbb{R}$, then z will satisfy $\left|\frac{z-1}{z+1}\right| = 1$; which describes a line in the plane, notably the imaginary axis.



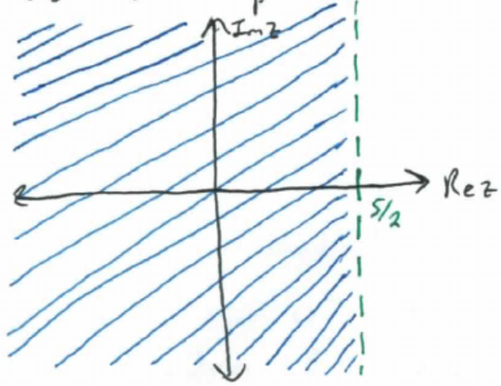
* thickness added for emphasis, in reality there is none.

(c). This set is the set of points $z \in \mathbb{C}$ such that the distance between z and 3 is strictly larger than the distance between z and 2. Consider $z = x + iy$ with $x, y \in \mathbb{R}$, then the following are equivalent.

$$|z-3| > |z-2|$$
$$(x-3)^2 + y^2 > (x-2)^2 + y^2$$
$$\tfrac{5}{2} > x \ .$$

Thus this set is $\left\{z \in \mathbb{C} \colon \operatorname{Re} z < \frac{5}{2}\right\}$. Such a set is commonly called a half-plane. This set is a region. Geometrically:

$z \in \mathbb{C}$ satisfying $|z-3| > |z-2|$ must satisfy $(x-3)^2 > (x-2)^2$, which reduces to $x < 5/2$ using the properties of $\mathbb{R}$. So if $x \in (-\infty, 5/2)$ and $y \in \mathbb{R}$, then z satisfies $|z-3| > |z-2|$. This describes a partition of the plane.



This is open and connected. Hence it <u>is</u> a region.

(d). This set is the set of points $z \in \mathbb{C}$ such that $z \neq 0$ and $z\,\overline{z} = 1$. Note that, for $z = x + iy$ with $x, y \in \mathbb{R}$,

$$z\,\overline{z} = (x + iy)\ (x - iy) = x^2 + y^2 = |z|^2\ .$$

Another approach would be to write $z = re^{i\theta}$ with $r, \theta \in \mathbb{R}$ and $r > 0$ and then compute

$$z\,\overline{z} = \left(re^{i\theta}\right)\ \left(\overline{re^{i\theta}}\right) = \left(re^{i\theta}\right)\ \left(\overline{r}\,\overline{e^{i\theta}}\right) = \left(re^{i\theta}\right)\ \left(re^{-i\theta}\right) = r^2 e^{i(\theta-\theta)} = r^2 e^0 = r^2 = |z|^2\ .$$

Thus this set

$$\{z \in \mathbb{C}\colon\ |z| = 1\}\ ,$$

that is, this set is the unit circle in the complex plane that is centered at the origin. This set is connected but is not open; thus, this set is not a region. Geometrically:

$z \in \mathbb{C}\setminus\{(0,0)\}$ satisfying $\frac{1}{z} = \overline{z}$ must satisfy $r^{-1} = r$, i.e. $r^2 = 1$. So $r = 1$ (since $r > 0$ by convention in polar form) and $\theta \in \mathbb{R}$ satisfies $z^{-1} = \overline{z}$. This describes ~~a etc~~ the unit circle.