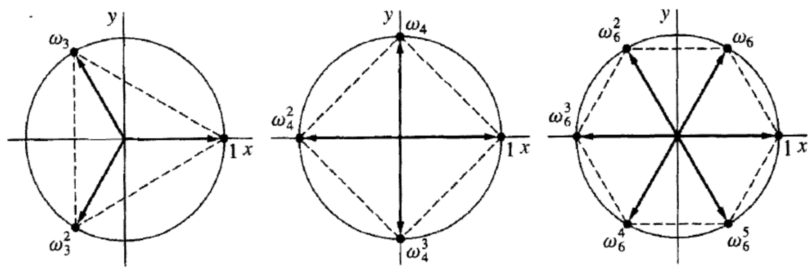
Definition. Following convention, for $n \in \mathbb{N}$, we set

$$\omega_n := \exp\left(i\,\frac{2\pi}{n}\right) \overset{\text{i.e.}}{=} e^{i\,\frac{2\pi}{n}}.$$

The n (distinct) <u>n^{th} roots of unity</u> (where ω_n^k denotes $(\omega_n)^k$) are

$$\left\{\ \omega_n^1\ ,\ \omega_n^2\ ,\ \ldots\ ,\ \omega_n^{n-1}\ ,\ \omega_n^n\left(\overset{\text{i.e}}{=}1\right)\ \right\}.$$

<u>Informative</u>. Compute, and draw on the unit circle, the n (distinct) n^{th} roots of unity for $n = 1, 2, 3, \ldots$ (for enough n's until you see the pattern ... below $n = 3, 4, 6$ are illustrated).



<u>Note</u>. Any n^{th} root of unity is a solution to the equation $z^n = 1$. Are there more? ⟨NO, as next Thm. shows.⟩
<u>Key Result</u>. The next theorem gives that, for ⟨fixed⟩ $r_0 > 0$ and $\theta_0 \in \mathbb{R}$, the solution set to the equation

$$z^n = r_0 e^{i\theta_0}$$

is the set (of n distinct elements)

$$\left\{\sqrt[n]{r_0}\ \left(e^{i\theta_0}\right)^{1/n}\ \left(e^{i2\pi k}\right)^{1/n}\ : k = 0, 1, 2, \ldots, n-1\right\}.$$

Theorem. *Let $n \in \mathbb{N}$ and $z_0 = r_0 e^{i\theta_0} \in \mathbb{C} \setminus \{0\}$ with $r_0 > 0$ and $\theta \in \mathbb{R}$.* ⟨so θ_0 is any element from *arg* z_0⟩. *Then the solution set of the equation*

$$z^n = z_0$$

is the set (of n <u>distinct</u> elements)

$$\left\{\sqrt[n]{r_0}\ \exp\left[\frac{i}{n}\left(\theta_0 + 2\pi k\right)\right] \in \mathbb{C}\colon k = 0, 1, 2, \ldots, n-1\right\} \overset{\text{i.e.}}{=} \left\{\sqrt[n]{|z_0|}\, e^{i\,\frac{\theta_0}{n}}\ (\omega_n)^k \in \mathbb{C}\colon k = 0, 1, 2, \ldots, n-1\right\}$$

where $w_n = e^{i\,\frac{2\pi}{n}}$. Furthermore, if $c \in \mathbb{C}$ is any solution to $z^n = z_0$, then

$$\left\{c\ ,\ c\omega_n^1\ ,\ c\omega_n^2\ ,\ c\omega_n^3\ ,\ \ldots\ ,\ c\omega_n^{n-1}\right\}$$

is a solution set to $z^n = z_0$. ⟨$c = c\omega_n^0$⟩. *If $-\pi < \theta_0 \leq \pi$* ⟨*i.e., θ_0 is the principal value of the argument of $r_0 e^{i\theta_0}$*⟩, *then $\sqrt[n]{r_0}\, e^{i\,\frac{\theta_0}{n}}$ is called the <u>principal n^{th} root of z_0</u>.*

Proof's key calculation. LTGBG. Then $\sqrt[n]{r_0} \in \mathbb{R}^{>0}$ since $r_0 \in \mathbb{R}^{>0}$. Then TFAE.

$$z_0 = z^n$$

$$r_0 e^{i\theta_0} = \left(re^{i\theta}\right)^n$$

$$r_0 e^{i(\theta_0 + 2\pi k)} = r^n e^{in\theta} \qquad \text{for any } k \in \mathbb{Z}$$

$$r = \sqrt[n]{r_0} \qquad \text{and} \qquad \theta = \frac{\theta_0}{n} + \frac{2\pi k}{n} \qquad \text{for any } k \in \mathbb{Z}$$

$$z = \sqrt[n]{r_0}\ \exp\left[i\left(\frac{\theta_0}{n} + \frac{2\pi k}{n}\right)\right] \qquad \text{for any } k \in \mathbb{Z}$$

$$z \in \left\{\sqrt[n]{r_0}\exp\left[i\left(\frac{\theta_0}{n} + \frac{2\pi k}{n}\right)\right] \in \mathbb{C}\colon k = 0, 1, 2, \ldots, n-1\right\}\ . \qquad (☺)$$

<u>Lesson</u>. Need to take care in taking n^{th} roots of complex numbers. The n^{th} roots of a (nonzero) complex number is a set with n distinct elements.
<u>Reference</u>. *Complex Variables and Appl.* by Brown and Churchill (Ch.1's §: *Roots of Complex Numbers*).

You are **strongly** encouraged to work in groups, following the procedure as in homework MS09.

Exercise pCA 2. Solve $z^2 - 4z + (4+2i) = 0$. Express your solution(s) in the form $a + ib$ with $a, b \in \mathbb{R}$.

First Solution. Note that TFAE.

$$\begin{aligned} z^2 - 4z + (4+2i) &= 0 \\ (z-2)^2 + 2i &= 0 \\ (z-2)^2 &= -2i \\ (z-2)^2 &= 2e^{\left(i\left(\frac{-\pi}{2}+2\pi k\right)\right)} \quad , \quad \text{for any } k \in \mathbb{Z} \\ z - 2 &= \sqrt{2}e^{\left(i\left(\frac{-\pi}{4}+\pi k\right)\right)} \quad , \quad \text{for any } k \in \mathbb{Z} \\ z &= 2 + \sqrt{2}e^{\left(i\left(\frac{-\pi}{4}+\pi k\right)\right)} \quad , \quad k = 0 \text{ or } k = 1 \ . \end{aligned}$$

Note

$$2 + \sqrt{2}e^{\left(i\left(\frac{-\pi}{4}+\pi k\right)\right)} = \begin{cases} z = 2 + \sqrt{2}e^{\left(i\left(\frac{-\pi}{4}\right)\right)} = 2 + \sqrt{2}\left(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right) = 3 - i & \text{if } k = 0 \\ z = 2 + \sqrt{2}e^{\left(i\left(\frac{3\pi}{4}\right)\right)} = 2 + \sqrt{2}\left(\frac{-1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = 1 + i & \text{if } k = 1. \end{cases}$$

Thus the solutions to $z^2 - 4z + (4+2i) = 0$ are $\boxed{3 - i \text{ and } 1 + i}$.

Second Solution. Consider the equation

$$az^2 + bz + c = 0 \tag{2.1}$$

where $a, b, c \in \mathbb{C}$ and $a \neq 0$. Simple algebra ⟨just as you did in grammer school to solve the quadratic equation with real coefficients, next complete the square⟩ gives that (2.1) holds if and only if

$$\left(z + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2} \ . \tag{2.2}$$

Case 1: $b^2 - 4ac = 0$. By (2.2), it is clear that (2.1) has one solution $z = \frac{-b}{2a} \overset{\text{note}}{=} \frac{-b \,\pm\, \sqrt{b^2-4ac}}{2a}$.

Case 2: $b^2 - 4ac \neq 0$. Let $d \in \mathbb{C}$ be any (complex) square root of $b^2 - 4ac$ ⟨so $d^2 = b^2 - 4ac$⟩. The Theorem on *Roots of Complex Numbers* gives that the solution set to the equation $z^2 = b^2 - 4ac$ is $\{d, d\omega_2\} = \left\{d, de^{\frac{i2\pi}{2}}\right\} = \{d, -d\}$. So by (2.2), the solution set to (2.1) is the 2 point set $\left\{\frac{-b \,\pm\, d}{2a}\right\}$.

Summarizing: The solution set to $az^2 + bz + c = 0$, where $a, b, c \in \mathbb{C}$ with $a \neq 0$, is

$$z \;=\; \frac{-b \;\pm\; \sqrt{b^2 - 4ac}}{2a} \tag{2.3}$$

WHERE the $\sqrt{b^2 - 4ac}$ in (2.3) denotes any (complex) square root of $b^2 - 4ac$.

Now consider the problem at hand $z^2 - 4z + (4+2i) = 0$. Here,

$$b^2 - 4ac = (-4)^2 - 4(1)(4+2i) = -8i = 8e^{i\left(\frac{-\pi}{2}\right)} \overset{\forall k \in \mathbb{Z}}{=} 8e^{i\left(\frac{-\pi}{2}+2\pi k\right)} \ . \tag{2.5}$$

A complex square root (take $k = 0$ in (2.5)) of $-8i$ is

$$\sqrt{8}\, e^{\frac{-i\pi}{4}} = 2\sqrt{2}\left(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right) = 2\,(1 - i) \ .$$

Note

$$\frac{4 \pm 2\,(1-i)}{2} = 2 \pm (1 - i) = \begin{cases} 3 - i & \text{for the plus} \\ 1 + i & \text{for the minus.} \end{cases}$$

Thus the solutions to $z^2 - 4z + (4+2i) = 0$ are $\boxed{3 - i \text{ and } 1 + i}$.