In 5a and 5b, display some of the statements for ease of reading,

[redacted]

Start a new paragraph with an indentation. (in Tex - just leave a blank line)

> You learn a lot talking math with others. Thus you are **strongly** encouraged to work in groups (up to size 17) on homework. A group is to come to an agreement of the finished paper. Over Blackboard, ONE group member (e.g., Bella) should submit the finished paper while each of the other group members (as so that I can return a commented graded paper to you) should just pull up the assignment on Blackboard and write a note in the white `Comment` box that, e.g., Bella submitted my paper.

Metric Space Exercise 5. Variant of 2.1.45.13 (p. 92).

See the (2 pages, updated, 2 due to questions in class) handout on Metric Spaces Basic Definitions.

In problems 5a-5d, you may use anything on this handout, from class lectures, and the book.

Metric Space Exercise 5a. Show that for all subsets A and B of a metric space X,

$$A^o \cap B^o = (A \cap B)^o .$$

Great use of the fact that S^o is the largest open set contained in S!!

Proof. First, $\cancel{[}(A \cap B)^o \subset A\cancel{]} \cancel{\wedge}$ and $\cancel{[}(A \cap B)^o \subset B\cancel{]}$, and by $(A \cap B)^o$ being open we have $\cancel{[}(A \cap B)^o \subset A^o\cancel{]} \cancel{\wedge}$ and $\cancel{[}(A \cap B)^o \subset B^o\cancel{]}$. Thus $\cancel{\Rightarrow}$ $(A \cap B)^o \subset A^o \cap B^o$.

Next, $[A^o \subset A] \cancel{\wedge}$ and $[B^o \subset B]$, so $\cancel{\Rightarrow}$ $A^o \cap B^o \subset A \cap B$, and by $A^o \cap B^o$ being open, as an intersection of two opens we get $A^o \cap B^o \subset (A \cap B)^o$, and we are done. $\square$

Metric Space Exercise 5b. Show that for all subsets A and B of a metric space X,

$$\overline{A} \cup \overline{B} = \overline{A \cup B}.$$

Great use of fact that $\overline{S}$ is the smallest closed set containing S!!

Proof. First, $\cancel{[}\overline{A} \supset A\cancel{]} \cancel{\wedge}$ and $\cancel{[}\overline{B} \supset B\cancel{]}$. So $\cancel{\Rightarrow}$ $\overline{A} \cup \overline{B} \supset A \cup B$, and by $\overline{A} \cup \overline{B}$ being closed as the union of two closeds, we get $\overline{A} \cup \overline{B} \supset \overline{A \cup B}$.

Next, $\cancel{[}\overline{A \cup B} \supset A\cancel{]} \cancel{\wedge}$ and $\cancel{[}\overline{A \cup B} \supset B\cancel{]}$, and by $\overline{A \cup B}$ being closed we have $\cancel{[}\overline{A \cup B} \supset \overline{A}\cancel{]} \cancel{\wedge}$ and $\cancel{[}\overline{A \cup B} \supset \overline{B}\cancel{]}$. So $\cancel{\Rightarrow}$ $\overline{A \cup B} \supset \overline{A} \cup \overline{B}$, and we are done. $\square$

✓ **Metric Space Exercise 5c.** Show by means of an example that, in general,

$$A^o \cup B^o \neq (A \cup B)^o .$$

Proof. Let $A = [-1, 0], B = [0, 1]$. We have $A^o = (-1, 0), B^o = (0, 1), (A \cup B)^o = (-1, 1)$, so

$$A^o \cup B^o = (-1, 0) \cup (0, 1) \neq (-1, 1) = (A \cup B)^o .$$

$\square$

✓ **Metric Space Exercise 5d.** Show by means of an example that, in general,

$$\overline{A} \cap \overline{B} \neq \overline{A \cap B}.$$

Proof. Take $A = [-1, 0), B = (0, 1]$, so that $\overline{A} = [-1, 0], \overline{B} = [0, 1], A \cap B = \emptyset$. We get:

$$\overline{A} \cap \overline{B} = \{0\} \neq \emptyset = \overline{A \cap B}.$$

run Tex one more time to make ??, turn into numbers.

[redacted]

Metric Space Exercise 5e. Let S be a subset of a metric space X. Show that

$$(S^o)^c = \overline{S^c}.$$

From Metric Spaces Basic Definitions, you may use anything up to, and including, (6i) on page 2. You may not use (7i) on page 2.

Proof. By definition, we have $S^o = \bigcup_{U\in\mathcal{U}} U$ where $\mathcal{U} = \{U \text{ is open in } X : U \subset S\}$. Taking complement both sides, we get

$$(S^o)^c = \bigcap_{U\in\mathcal{U}} U^c$$

where the intersection is taken over all open sets U such that $U \subset S$, the latter inclusion holds if and only if $S^c \subset U^c$. In other words, we can write

$$(S^o)^c = \bigcap_{U^c\in\mathcal{U}'} U^c$$

where $\mathcal{U}' = \{U^c \text{ is closed in } X : S^c \subset U^c\}$.

Let us denote U^c by F then $\mathcal{U}' = \{F \text{ is closed in } X : S^c \subset F\}$. Then we have

$$(S^o)^c = \bigcup_{F\in\mathcal{U}'} F$$

by definition of closure of a set, it is the set $\overline{S^c}$. This shows the desired. □ ✓

nicely done!