You learn a lot talking math with others. Thus you are **strongly** encouraged to work in groups (up to size 17) on homework. A group is to come to an agreement of the finished paper. Over Blackboard, ONE group member (e.g., Bella) should submit the finished paper while each of the other group members (as so that I can return a commented graded paper to you) should just pull up the assignment on Blackboard and write a note in the white `Comment` box that, e.g., Bella submitted my paper.

Throughout this Exercise, I is a nonempty closed bounded interval of $\mathbb{R}$.

Since I is a closed bounded interval, $C(I,\mathbb{R}) = \mathscr{C}(I,\mathbb{R}) \subset \mathscr{R}(I,\mathbb{R})$ where:

$$C(I,\mathbb{R}) := \{f\colon I \to \mathbb{R} \mid f \text{ is continuous on } I\}.$$

$$\mathscr{C}(I,\mathbb{R}) := \{f\colon I \to \mathbb{R} \mid f \text{ is bounded and continuous on } I\}$$

$$\mathscr{R}(I,\mathbb{R}) := \{f\colon I \to \mathbb{R} \mid f \text{ is Riemann integrable over } I\}.$$

Note that $C(I,\mathbb{R})$ (resp. $\mathscr{C}(I,\mathbb{R})$, $\mathscr{R}(I,\mathbb{R})$) is a *vector space* over the field $\mathbb{R}$.

Now let $1 \le p < \infty$. If $h \in C(I,\mathbb{R})$ then $|h|^p \in C(I,\mathbb{R}) \subset \mathscr{R}(I,\mathbb{R})$. Thus we can define the function $\|\cdot\|_p : C(I,\mathbb{R}) \to \mathbb{R}$ by

$$\|f\|_p := \left[\int_I |f(t)|^p\right]^{1/p} \quad \text{, for } f \in C(I,\mathbb{R})$$

and corresponding function $d_p\colon C(I,\mathbb{R}) \times C(I,\mathbb{R}) \to \mathbb{R}$ by

$$d_p(f,g) := \left[\int_I |f(t) - g(t)|^p\right]^{1/p} \quad \text{, for } f,g \in C(I,\mathbb{R})$$

Note $d_p(f,g) = \|f-g\|_p$.

Recall that $\|\cdot\| : X \to \mathbb{R}$ is a <u>norm</u> on a vector space X (over the field $\mathbb{K}$ of $\mathbb{R}$ or $\mathbb{C}$) provided

(N1) $\|x\| = 0$ if and only if $x = 0$,

(N2) $\|\alpha x\| = |\alpha|\,\|x\|$,

(N3) $\|x+y\| \le \|x\| + \|y\|$

for each $x, y \in X$ and $\alpha \in \mathbb{K}$.

Metric Space Exercise 4. Variant of 2.1.45.2 (p. 90).

This is the *function* version of the *sequence* Exercise 2.1.45.1 (hwMS2). You may use, without (re)proving, anything from hwMS2.

Metric Space Exercise 4a. Let $\|\cdot\| : X \to \mathbb{R}$ is a norm on a vector space X (over $\mathbb{R}$ or $\mathbb{C}$).

Let $f \in C(I,\mathbb{R})$ and $\alpha \in \mathbb{R}$. Let $1 \le p < \infty$.

Show that $\|x\| \ge 0$ for all $x \in X$ by using properties (N1), (N2), and (N3).

Show that $\|f\|_p = 0$ if and only if $f = 0 1_I$ $\langle$i.e., if and only if $f(t) = 0$ for all $t \in I\rangle$.

Show that $\|\alpha f\|_p = |\alpha|\,\|f\|_p$.

Proof. Let $\|\cdot\| : X \to \mathbb{R}$ is a norm on a vector space X (over $\mathbb{R}$ or $\mathbb{C}$). Let $f \in C(I, \mathbb{R})$ and $\alpha \in \mathbb{R}$. Let $1 \leq p < \infty$.

✓ **4a(i)**

Suppose $x \in X$. We observe that,

$$\begin{aligned} & \|x + (-x)\| \leq \|x\| + \|-x\| && \text{(N3)} \\ \Leftrightarrow \quad & \|0\| \leq \|x\| + \|(-1)\,x\| \\ \Leftrightarrow \quad & 0 \leq \|x\| + |-1| \cdot \|x\| && \text{(N1), (N2)} \\ \Leftrightarrow \quad & 0 \leq 2\,\|x\| \\ \Leftrightarrow \quad & 0 \leq \|x\|\,. \end{aligned}$$

Therefore, we conclude that $\|x\| \geq 0$ for all $x \in X$.

4a(ii)

Suppose $\|f\|_p = 0$. We observe that,

$$\begin{aligned} & \|f\|_p = 0 \\ \Leftrightarrow \quad & \left(\int_I |f(t)|^p\right)^{\frac{1}{p}} = 0 \\ \Leftrightarrow \quad & \int_I |f(t)|^p = 0. \end{aligned}$$

Since $|f(t)| \geq 0$ for all $t \in I$, it follows that, since f is continuous,

$$\begin{aligned} & |f(t)|^{p} = 0 && \forall t \in I \\ \Leftrightarrow \quad & f(t) = 0. && \forall t \in I\,. \end{aligned}$$

Now, suppose that $f = 01_I$. Clearly,

$$\begin{aligned} & f(t) = 0 && \forall t \in I \\ \Rightarrow \quad & \left(\int_I |f(t)|^p\right)^{\frac{1}{p}} = 0 \\ \Leftrightarrow \quad & \|f\|_p = 0. \end{aligned}$$

Hence, we conclude that $\|f\|_p = 0$ if and only if $f = 01_I$.

4a(iii)

We observe that,

$$\begin{aligned}\|\alpha f\|_p &= \left(\int_I |\alpha f(t)|^p\right)^{\frac{1}{p}} \\ &= \left(\int_I |\alpha|^p\,|f(t)|^p\right)^{\frac{1}{p}} \\ &= |\alpha|\left(\int_I |f(t)|^p\right)^{\frac{1}{p}} \\ &= |\alpha|\,\|f\|_p\,.\end{aligned}$$

Thus, we conclude that $\|\alpha f\|_p = |\alpha|\,\|f\|_p$.

□

Metric Space Exercise 4b. (Hölder's Inequality for Continuous Functions). Let $1 < p < \infty$. Let $f, g \in C(I, \mathbb{R})$ and $p' \in (1, \infty)$ be the *conjugate exponent* of p (thus $\frac{1}{p} + \frac{1}{p'} = 1$). Show

$$\|fg\|_1 \leq \|f\|_p\,\|g\|_{p'} \tag{H}$$

that is

$$\int_I |f(t)\,g(t)|\;dt \leq \left[\int_I |f(t)|^p\;dt\right]^{1/p}\left[\int_I |g(t)|^{p'}\;dt\right]^{1/p'}.$$

Proof. Let $1 < p < \infty$. Suppose that $f(t) \neq 0 1_I$ and $g(t) \neq 0 1_I$; for other wise, ~~Otherwise,~~ Hölder's Inequality is trivially true. Utilizing Young's inequality point wise, we observe that,

$$\begin{aligned}\frac{\int_I |f(t)g(t)|\,dt}{\left[\int_I |f(t)|^p\,dt\right]^{\frac{1}{p}}\left[\int_I |g(t)|^{p'}\,dt\right]^{\frac{1}{p'}}} &= \int_I \frac{|f(t)|}{\left[\int_I |f(t)|^p\,dt\right]^{\frac{1}{p}}} \cdot \frac{|g(t)|}{\left[\int_I |g(t)|^{p'}\,dt\right]^{\frac{1}{p'}}}\;dt \\ &\leq \int_I \frac{|f(t)|^p}{p\int_I |f(t)|^p} + \frac{|g(t)|^{p'}}{p'\int_I |g(t)|^{p'}}\;dt \\ &= \frac{1}{p}\cdot\frac{\int_I |f(t)|^p\,dt}{\int_I |f(t)|^p\,dt} + \frac{1}{p'}\cdot\frac{\int_I |g(t)|^{p'}\,dt}{\int_I |g(t)|^{p'}\,dt} \\ &= \frac{1}{p} + \frac{1}{p'} = 1.\end{aligned}$$

do you see you are using Young Ineq. pointwise here?

Hence,

$$\int_I |f(t)g(t)dt| \leq \left[\int_I |f(t)|^p\,dt\right]^{1/p}\cdot\left[\int_I |g(t)|^{p'}\,dt\right]^{1/p'}.$$

Thus,

$$\|fg\|_1 \leq \|f\|_p\,\|g\|_{p'}\,.$$

□

[redacted]

Metric Space Exercise 4c. (Minkowski's Inequality for Cnt. Functions). Let $1 < p < \infty$.

Let $f, g \in C(I, \mathbb{R})$. Show

$$\|f+g\|_p \leq \|f\|_p + \|g\|_p \tag{M}$$

that is

$$\left[\int_I |f(t)+g(t)|^p \, dt\right]^{1/p} \leq \left[\int_I |f(t)|^p \, dt\right]^{1/p} + \left[\int_I |g(t)|^p \, dt\right]^{1/p}.$$

Does Minkowski's Inequality hold when $p = 1$?

Hint for $1 < p < \infty$. If $a, b \in \mathbb{R}$ and $1 < p < \infty$, then $0 \neq p - 1$ $\langle$so $|0+0|^{p-1}$ makes sense$\rangle$ and so

$$\begin{aligned} |a+b|^p &= |a+b|^1 \, |a+b|^{p-1} \\ &\leq [\ |a| + |b|\ ]\ |a+b|^{p-1} \\ &= |a|\ \ |a+b|^{p-1} + |b|\ \ |a+b|^{p-1}. \end{aligned}$$

Proof. Let $1 \leq p < \infty$. Suppose $f(t) \neq 0 1_I$ and $g(t) \neq 0 1_I$~~. Otherwise~~ for otherwise, Minkowski's Inequality is trivially true. If $p = 1$, Minkowski's Inequality holds by the Triangle Inequality. good So now let $1<p<\infty$. We observe that,

$$\begin{aligned} \int_I |f(t)+g(t)|^p \, dt &= \int_I |f(t)+g(t)| \, |f(t)+g(t)|^{p-1} \, dt \\ &\leq \int_I \big(\, |f(t)| + |g(t)| \,\big) \, |f(t)+g(t)|^{p-1} \, dt \\ &= \int_I |f(t)| \, |f(t)+g(t)|^{p-1} \, dt + \int_I |g(t)| \, |f(t)+g(t)|^{p-1} \, dt. \\ &= \int_I \left| [f(t)] \, [f(t)+g(t)]^{p-1} \right| dt + \int_I \left| [g(t)] \, [f(t)+g(t)]^{p-1} \right| dt. \end{aligned}$$

We apply Hölder's inequality and since $p' = \dfrac{p}{p-1}$ we see that,

$$\begin{aligned} \int_I |f(t)+g(t)|^p \, dt &\leq \int_I \left| [f(t)] \, [f(t)+g(t)]^{p-1} \right| dt + \int_I \left| [g(t)] \, [f(t)+g(t)]^{p-1} \right| dt. \\ &\leq \left[\int_I |f(t)|^p \, dt\right]^{\frac{1}{p}} \left[\int_I |f(t)+g(t)|^{(p-1)p'} \, dt\right]^{\frac{1}{p'}} + \left[\int_I |g(t)|^p \, dt\right]^{\frac{1}{p}} \left[\int_I |f(t)+g(t)|^{(p-1)p'} \, dt\right]^{\frac{1}{p'}} \\ &= \left[\int_I |f(t)|^p \, dt\right]^{\frac{1}{p}} \left[\int_I |f(t)+g(t)|^p \, dt\right]^{\frac{p-1}{p}} + \left[\int_I |g(t)|^p \, dt\right]^{\frac{1}{p}} \left[\int_I |f(t)+g(t)|^p \, dt\right]^{\frac{p-1}{p}} \\ &= \left(\left[\int_I |f(t)|^p \, dt\right]^{\frac{1}{p}} + \left[\int_I |g(t)|^p \, dt\right]^{\frac{1}{p}}\right) \left[\int_I |f(t)+g(t)|^p \, dt\right]^{\frac{p-1}{p}}. \end{aligned}$$

Hence,

$$\left[\int_I |f(t)+g(t)|^p \, dt\right]^{\frac{1}{p}} \leq \left[\int_I |f(t)|^p \, dt\right]^{\frac{1}{p}} + \left[\int_I |g(t)|^p \, dt\right]^{\frac{1}{p}}.$$

good

Thus,

$$\|f+g\|_p \leq \|f\|_p + \|g\|_p.$$

✓

$\square$

Metric Space Exercise 4d. Let $1 \leq p \leq \infty$.

Conclude that $\left(C(I,\mathbb{R}), \|\cdot\|_p\right)$ is a normed vector space.

Conclude that $(C(I,\mathbb{R}), d_p)$ is a metric space.

Proof. Let $1 \leq p \leq \infty$. In order for $\left(C(I,\mathbb{R}), \|\cdot\|_p\right)$ to be a normed vector space, it must satisfy the following properties:

(N1) $\|x\| = 0$ if and only if $x = 0$

For $1 \leq p < \infty$ we refer to **Metric Space Exercise 4a.**

✓ For $p = \infty$, we observe that,

$$\|f\|_\infty = \sup_{t\in I} |f(t)| = 0 \Leftrightarrow |f(t)| = 0 \Leftrightarrow f(t) = 0 \text{ for every } t \in I.$$

(N2) $\|\alpha x\| = |\alpha| \|x\|$

For $1 \leq p < \infty$ we refer to **Metric Space Exercise 4a.**

✓ For $p = \infty$, we observe that,

$$\|\alpha f\|_\infty = \sup_{t\in I} |\alpha f(t)| = \sup_{t\in I} |\alpha| |f(t)| = |\alpha| \sup_{t\in I} |f(t)| = |\alpha| \|f\|_\infty .$$

(N3) $\|x + y\| \leq \|x\| + \|y\|$

For $1 \leq p < \infty$, we see that Minkowski's Inequality satisfies this property.

For $p = \infty$, we observe that, good !

$$\|f + g\|_\infty = \sup_{t\in I} |f(t) + g(t)| \leq \sup_{t\in I} |f(t)| + \sup_{t\in I} |g(t)| = \|f\|_\infty + \|g\|_\infty .$$

Since $\left(C(I,\mathbb{R}), \|\cdot\|_p\right)$ is a normed vector space, we define $d_p(f,g) := \|f - g\|_p$. Thus, d_p satisfies the required properties and we conclude that $(C(I,\mathbb{R}), d_p)$ is a metric space. $\square$