Start a new ¶ with an indentation. To do this in tex, just skip a line.

You learn a lot talking math with others. Thus you are **strongly** encouraged to work in groups (up to size 17) on homework. A group is to come to an agreement of the finished paper and then each group member should submit over Blackboard the identical finished paper. Follow the instructions at the top of the LaTex file to but all PINs and Names on the paper. A graded copy of the group's finished paper will be returned to each group member.

Metric Space Exercise 3. Variant of 2.1.45.4 (p. 90)

Let S be the set of all sequences of real numbers and define $d\colon S \times S \to \mathbb{R}$ by

$$d\,(x,y) := \sum_{n=1}^{\infty} \frac{|x_n - y_n|}{2^n\,[1+|x_n-y_n|]} \;\;,\;\; x = \{x_n\}_{n=1}^{\infty} \in S \text{ and } y = \{y_n\}_{n=1}^{\infty} \in S. \tag{1}$$

Show that (S,d) is a metric space.

Hint: you may use, without proving, that the function $f\colon\, [0,\infty) \to \mathbb{R}$ given by $f\,(t) := \dfrac{t}{1+t}$ is a strictly increasing function.

LTGBG.

Proof. First we show that $d(x,y) \geq 0$, for all $x,y \in S$. Note that each term in the sequence of partial sums $\left\{\sum_{n=1}^{N} \frac{|x_n-y_n|}{2^n[1+|x_n-y_n|]}\right\}_{N\geq 1}$ is non-negative. Since

$$d(x,y) = \lim_{N\to\infty} \sum_{n=1}^{N} \frac{|x_n-y_n|}{2^n\,[1+|x_n-y_n|]}, \tag{2}$$

the limit, $d(x,y)$ is non-negative as well.

Now if we have $x = y$, then for each $n \in \mathbb{N}$, $x_n = y_n$. So each term of the infinite series in 1 is zero, hence the sum $d(x,y)$ is also zero. Conversely let us assume that $d(x,y) = 0$. Since the sequence $\left\{\sum_{n=1}^{N} \frac{|x_n-y_n|}{2^n[1+|x_n-y_n|]}\right\}_{N\geq 1}$ is monotonically increasing with limit zero, each partial sum must be zero. Thus each term $\frac{|x_n-y_n|}{2^n[1+|x_n-y_n|]}$ is also zero. Hence $x_n = y_n$ for each $n \in \mathbb{N}$, giving $x=y$.

Moreover since $|x_n - y_n| \;=\; |y_n - x_n|$ for each $n \in \mathbb{N}$, we have $d(x,y) = d(y,x)$.

Finally to see the triangle inequality, we first note that the function $f(t) = \frac{t}{1+t}$ is strictly increasing on the interval $[0,\infty)$. For each $n \in \mathbb{N}$, we have

$$|x_n - z_n| \leq |x_n - y_n| + |y_n - z_n|$$

Applying f on both sides gives

$$\begin{aligned}\frac{|x_n - z_n|}{1 + |x_n - z_n|} &\leq \frac{|x_n - y_n| + |y_n - z_n|}{1 + |x_n - y_n| + |y_n - z_n|} \\ &= \frac{|x_n - y_n|}{1 + |x_n - y_n| + |y_n - z_n|} + \frac{|y_n - z_n|}{1 + |x_n - y_n| + |y_n - z_n|} \\ &\leq \frac{|x_n - y_n|}{1 + |x_n - y_n|} + \frac{|y_n - z_n|}{1 + |y_n - z_n|}.\end{aligned}$$

Multiplying both sides by $\frac{1}{2^n}$, and taking sum over n from 1 to N gives

$$\sum_{n=1}^{N} \frac{|x_n - z_n|}{2^n[1 + |x_n - z_n|]} \ \leq \ \sum_{n=1}^{N} \frac{|x_n - y_n|}{2^n[1 + |x_n - y_n|]} \ + \ \sum_{n=1}^{N} \frac{|y_n - z_n|}{2^n[1 + |y_n - z_n|]}.$$

Letting $N \to \infty$ gives us the desired inequality

$$\sum_{n=1}^{\infty} \frac{|x_n - z_n|}{2^n[1 + |x_n - z_n|]} \ \leq \ \sum_{n=1}^{\infty} \frac{|x_n - y_n|}{2^n[1 + |x_n - y_n|]} \ + \ \sum_{n=1}^{\infty} \frac{|y_n - z_n|}{2^n[1 + |y_n - z_n|]},$$

that is

$$d(x,z) \ \leq \ d(x,y) \ + \ d(y,z).$$

This shows that d is a metric on set S, making (S,d) a metric space. □

nice!