[redacted]

You learn a lot talking math with others. Thus you are **strongly** encouraged to work in groups (up to size 17) on homework. A group is to come to an agreement of the finished paper and then each group member should submit over Blackboard the identical finished paper. Follow the instructions at the top of the LaTex file to but all PINs and Names on the paper. A graded copy of the group's finished paper will be returned to each group member.

Let $N \in \mathbb{N}$. Define $d_p \colon \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ by

$$d_p\left(\{x_i\}_{i=1}^N, \{y_i\}_{i=1}^N\right) := \begin{cases} \left[\sum_{i=1}^N |x_i - y_1|^p\right]^{\frac{1}{p}} & \text{, if } 1 \le p < \infty \\ \sup_{1 \le i \le N} |x_i - y_i| & \text{, if } p = \infty \end{cases} \tag{1}$$

and $\|\cdot\|_{\ell_p} : \mathbb{R}^N \to \mathbb{R}$ by

$$\left\|\{x_i\}_{i=1}^N\right\|_{\ell_p} := \begin{cases} \left[\sum_{i=1}^N |x_i|^p\right]^{\frac{1}{p}} & \text{, if } 1 \le p < \infty \\ \sup_{1 \le i \le N} |x_i| & \text{, if } p = \infty \end{cases} \tag{2}$$

for $x := \{x_i\}_{i=1}^N$ and $y := \{y_i\}_{i=1}^N$ in $\mathbb{R}^N$. Note $d_p(x, y) = \|x - y\|_{\ell_p}$ where $\{x_i\}_{i=1}^N - \{y_i\}_{i=1}^N := \{x_i - y_i\}_{i=1}^N$. In class we observed that (R^N, d_p) is a metric space when p is 1 or ∞. The goal is this exercies is to extend this fact to $1 \le p \le \infty$. We shall use the concept of convex functions, which is useful in showing many inequalities in mathematics. This Exercise set is a variant of Exercise 2.1.45.1 (page 89). The actual statements of the exercise's parts are toward the end of the file.

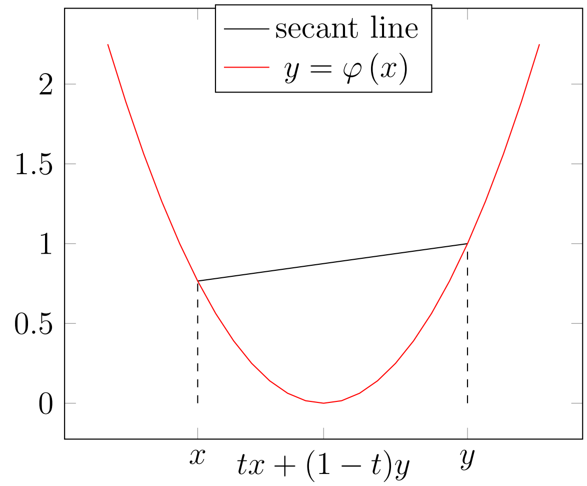
Notation 1. Throughout this Exercise, $N \in \mathbb{N}$ is fixed and

- $(a, b) = I \subset \mathbb{R}$ where $a, b \in \mathbb{R} \cup \{\pm\infty\}$
- $\varphi \colon I \to \mathbb{R}$ is a function
- $p \in (1, \infty)$ and its *conjugate exponent* $p' \in (1, \infty)$ is defined by $\frac{1}{p} + \frac{1}{p'} = 1$.

Definition 2. The function $\varphi \colon I \to \mathbb{R}$ is *convex* provided

$$[x, y \in I \text{ and } t \in (0,1)] \implies \varphi(tx + (1-t)y) \le t\varphi(x) + (1-t)\varphi(y)\,. \tag{3}$$

Graphically, for a convex function φ on I and any $(x, y) \subset I$, on the interval (x, y), the graph of the function φ on the interval (x, y) lie below the secant line through $(x, \varphi(x))$ and $(y, \varphi(y))$.



Proposition 3. *Let the second derivative of $f\colon I \to \mathbb{R}$ exist at each point of I. Then f is a convex function on I if and only if $f''(x) \geq 0$ for each $x \in I$.*

A proof of Proposition 3 is posted on the handout page.

Lemma 4. *Let $\varphi\colon I \to \mathbb{R}$ be convex. Then:*

$$x_i \in I \quad , \quad t_i \in (0,1) \quad , \quad \sum_{i=1}^{n} t_i = 1 \quad \Longrightarrow \quad \varphi\left(\sum_{i=1}^{n} t_i x_i\right) \leq \sum_{i=1}^{n} t_i \, \varphi(x_i) \tag{4}$$

Lemma 4 follows easily from induction and the definition of convex function.

Lemma 5.

$$x_i \in I \quad , \quad t_i \in (0,1) \quad \Longrightarrow \quad \varphi\left(\frac{\sum_{i=1}^{n} t_i x_i}{\sum_{i=1}^{n} t_i}\right) \leq \frac{\sum_{i=1}^{n} t_i \, \varphi(x_i)}{\sum_{i=1}^{n} t_i} \ . \tag{5}$$

To see Lemma 5, let $\tilde{t}_i = \dfrac{t_i}{\sum_{j=1}^{n} t_j}$, note that $\sum_{i=1}^{n} \tilde{t}_i = 1$, and then use Lemma 4.

Recall 6. Geometric-Arithmetic Mean Inequality (GAM Inequality) $\langle \mathrm{GM} \leq \mathrm{AM} \rangle$

$$x_i \geq 0 \quad , \quad n \in \mathbb{N} \quad \Longrightarrow \quad \left(\prod_{i=1}^{n} x_i\right)^{1/n} \leq \frac{1}{n} \sum_{i=1}^{n} x_i \ .$$

The GAM Inequality follows directly for the Generalized Geometric-Arithmetic Mean Inequality.

Proposition 7 (Generalized Geometric-Arithmetic Mean Inequality)**.**

$$x_i \geq 0 \quad , \quad t_i \in (0,1) \quad , \quad \sum_{i=1}^{n} t_i = 1 \quad \Longrightarrow \quad \prod_{i=1}^{n} x_i^{t_i} \leq \sum_{i=1}^{n} t_i \, x_i \ . \tag{6}$$

The proof of Prop. 7 is part of your homework. See towards the end of this file for further deatils.

Proposition 8 (Young's Inequality)**.** *Let $1 < p < \infty$.*

$$a_i \geq 0 \quad , \quad 1 < p < \infty \qquad \Longrightarrow \qquad a_1 \quad \cdot a_2 \;\leq\; \frac{(a_1)^p}{p} \;+\; \frac{(a_2)^{p'}}{p'} \;. \tag{7}$$

The proof of Prop. 8 is part of your homework. See towards the end of this file for further deatils.

Theorem 9 (Hölder's Inequality in ℓ_p)**.** *Let $1 < p < \infty$.*
For sequences $\{x_i\}_{i=1}^N$ and $\{y_i\}_{i=1}^N$ from $\mathbb{R}$,

$$\left\|\{x_i \cdot y_i\}_{i=1}^N\right\|_{\ell_1} \;\leq\; \left\|\{x_i\}_{i=1}^N\right\|_{\ell_p} \quad\cdot\quad \left\|\{y_i\}_{i=1}^N\right\|_{\ell_{p'}} \tag{8}$$

that is

$$\sum_{i=1}^N |x_i y_i| \;\leq\; \left[\sum_{i=1}^N |x_i|^p\right]^{1/p} \quad\cdot\quad \left[\sum_{i=1}^N |y_i|^{p'}\right]^{1/p'} \;. \tag{9}$$

Note that when $p = 2 = p'$, Hölder's inequality is just the Cauchy-Schwarz inequality. The proof of Theorem 9 is part of your homework. See towards the end of this file for further deatils.

Theorem 10 (Minkowski's Inequality in ℓ_p)**.** *Let $1 < p < \infty$.*
For sequences $\{x_i\}_{i=1}^N$ and $\{y_i\}_{i=1}^N$ from $\mathbb{R}$,

$$\left\|\{x_i + y_i\}_{i=1}^N\right\|_{\ell_p} \;\leq\; \left\|\{x_i\}_{i=1}^N\right\|_{\ell_p} \;+\; \left\|\{y_i\}_{i=1}^N\right\|_{\ell_p} \tag{10}$$

that is

$$\left[\sum_{i=1}^N |x_i + y_i|^p\right]^{1/p} \;\leq\; \left[\sum_{i=1}^N |x_i|^p\right]^{1/p} \;+\; \left[\sum_{i=1}^N |y_i|^p\right]^{1/p} \;. \tag{11}$$

The proof of Thm. 10 is part of your homework. See towards the end of this file for further deatils.

Statement of Exercise's Parts a – e.

Metric Space Exercise 2a. Give a really short proof of Proposition 7 (Generalized GAM inequality) by applying Lemma 4 to a cleverly chosen convex function $y = \varphi(x)$. Be sure to carefully justify that your φ is convex.

Proof. LTGBG. Note firstly that if $x_i = 0$ for any $1 \leq i \leq n$, then the Generalized GAM inequality holds since the product will be zero whilst the sum will contain strictly nonnegative terms. We therefore take $x_i > 0$ for all i and select $y = \varphi(x)$ to be $y = -\ln(x)$. We claim that $\varphi : (0, \infty) \to \mathbb{R}$ is convex. To justify this, observe that $\varphi''(x) = \frac{1}{x^2} > 0$ for all $x \in (0, \infty)$. By Proposition 3, $\varphi(x) = -\ln(x)$ is convex. Now we will examine $-\ln\left(\prod_{i=1}^{n} x_i^{t_i}\right)$. Using properties of logarithms and the convexity of $-\ln(x)$, we have:

$$-\ln\left(\prod_{i=1}^{n} x_i^{t_i}\right) = -\sum_{i=1}^{n} t_i \ln(x_i) \geq -\ln\left(\sum_{i=1}^{n} t_i x_i\right) \quad \Rightarrow \quad \ln\left(\prod_{i=1}^{n} x_i^{t_i}\right) \leq \ln\left(\sum_{i=1}^{n} t_i x_i\right).$$

The Generalized GAM inequality follows immediately by exponentiating both sides of the last inequality using base e, completing the proof. □

Metric Space Exercise 2b. Give a really short proof of Proposition 8 (Young's inequality) using the Generalized GAM inequality (Proposition 7).

Proof. LTGBG. Recalling that $\frac{1}{p} + \frac{1}{p'} = 1$, we may invoke the Generalized GAM inequality in the following calculation which obtains Young's Inequality:

$$a_1 \cdot a_2 = (a_1^p)^{\frac{1}{p}} \cdot (a_2^{p'})^{\frac{1}{p'}} \leq \frac{1}{p}(a_1^p) + \frac{1}{p'}(a_2^{p'}) = \frac{(a_1)^p}{p} + \frac{(a_2)^{p'}}{p'}.$$

□

Metric Space Exercise 2c. Prove Theorem 9 (Hölder's Inequality) using Young's inequality (Proposition 8).

Proof. LTGBG. First of all, if $\sum_{i=1}^{N} |x_i|^p = 0$, then clearly $x_i = 0$ for all $1 \leq i \leq N$. Thus, in this case, Hölder's Inequality will hold because:

$$\sum_{i=1}^{N} |0 \cdot y_i| = 0 \leq 0 \cdot \left[\sum_{i=1}^{N} |y_i|^{p'}\right]^{\frac{1}{p'}}.$$

An analogous statement holds if $\sum_{i=1}^{N} |y_i|^{p'} = 0$, so we assume that neither sum is zero. Define sequences $\{v_i\}_{i=1}^N$ and $\{w_i\}_{i=1}^N$ by:

$$\{v_i\}_{i=1}^N = \frac{1}{\|\{x_i\}\|_{\ell_p}}\left(\{x_i\}_{i=1}^N\right) \quad \text{and} \quad \{w_i\}_{i=1}^N = \frac{1}{\|\{y_i\}\|_{\ell_{p'}}}\left(\{y_i\}_{i=1}^N\right).$$

Observe by their definitions that $\|\{v_i\}\|_{\ell_p} = \|\{w_i\}\|_{\ell_{p'}} = 1$. Next, let $a_1 = \frac{|v_j|}{\|\{v_i\}\|_{\ell_p}}$ and $a_2 = \frac{|w_j|}{\|\{w_i\}\|_{\ell_{p'}}}$ where $1 \leq j \leq N$. Then Young's Inequality assures us for each j that:

$$\frac{|v_j| \cdot |w_j|}{\|\{v_i\}\|_{\ell_p} \cdot \|\{w_i\}\|_{\ell_{p'}}} \leq \frac{|v_j|^p}{p\|\{v_i\}\|_{\ell_p}^p} + \frac{|w_j|^{p'}}{p'\|\{w_i\}\|_{\ell_{p'}}^{p'}} \quad \Rightarrow \quad |v_j \cdot w_j| \leq \frac{1}{p}|v_j|^p + \frac{1}{p'}|w_j|^{p'}.$$

Now we sum both sides of our inequality over j to obtain:

$$\sum_{j=1}^{N} |v_j \cdot w_j| \leq \sum_{j=1}^{N} \left(\frac{1}{p}|v_j|^p + \frac{1}{p'}|w_j|^{p'}\right) = \frac{1}{p}\sum_{j=1}^{N} |v_j|^p + \frac{1}{p'}\sum_{j=1}^{N} |w_j|^{p'} = \frac{1}{p}(1)^p + \frac{1}{p'}(1)^{p'} = 1.$$

This is to say that $\|\{v_j \cdot w_j\}\|_{\ell_1} \leq 1 = \|\{v_j\}\|_{\ell_p} \cdot \|\{w_j\}\|_{\ell_{p'}}$, which is Hölder's Inequality when the sequences have norm equal to 1. To extend this to $\{x_i\}_{i=1}^N$ and $\{y_i\}_{i=1}^N$, we compute:

$$\|\{x_i \cdot y_i\}\|_{\ell_1} = \sum_{i=1}^{N} \left(\|\{x_i\}\|_{\ell_p} \cdot |v_i|\right) \cdot \left(\|\{y_i\}\|_{\ell_{p'}} \cdot |w_i|\right) = \|\{x_i\}\|_{\ell_p} \cdot \|\{y_i\}\|_{\ell_{p'}} \cdot \sum_{i=1}^{N} |v_i| \cdot |w_i|.$$

We've shown that $\|\{x_i \cdot y_i\}\|_{\ell_1} = \|\{x_i\}\|_{\ell_p} \cdot \|\{y_i\}\|_{\ell_{p'}} \cdot \|\{v_j \cdot w_j\}\|_{\ell_1}$, but our earlier deduction that $\|\{v_j \cdot w_j\}\|_{\ell_1} \leq 1$ yields the desired conclusion, namely $\|\{x_i \cdot y_i\}\|_{\ell_1} \leq \|\{x_i\}\|_{\ell_p} \cdot \|\{y_i\}\|_{\ell_{p'}}$.

□

Metric Space Exercise 2d. Prove Theorem 10 (Minkowski's Inequality) using Hölder's inequality (Theorem 9).

Proof. LTGBG. As in the proof of Hölder's Inequality, it is easy to see that Minkowski's Inequality will hold when either $\sum_{i=1}^{N} |x_i|^p = 0$ or $\sum_{i=1}^{N} |y_i|^p = 0$, so let us take both sums to be nonzero. We manipulate the left-hand-side of (11) sans the $\frac{1}{p}$ power and use the triangle inequality:

$$\sum_{i=1}^{N} |x_i + y_i|^p = \sum_{i=1}^{N} |x_i + y_i| \cdot |x_i + y_i|^{p-1} \leq \sum_{i=1}^{N} \left(|x_i| + |y_i|\right) \cdot |x_i + y_i|^{p-1}.$$

Distributing, we reach:

$$\sum_{i=1}^{N} |x_i + y_i|^p \leq \sum_{i=1}^{N} |x_i| \cdot |x_i - y_i|^{p-1} + \sum_{i=1}^{N} |y_i| \cdot |x_i - y_i|^{p-1}.$$

Now we will apply Hölder's Inequality to both sums on the right-hand-side:

$$\sum_{i=1}^{N} |x_i + y_i|^p \leq \left[\sum_{i=1}^{N} |x_i|^p\right]^{\frac{1}{p}} \left[\sum_{i=1}^{N} \left(|x_i + y_i|^{(p-1)}\right)^{p'}\right]^{\frac{1}{p'}} + \left[\sum_{i=1}^{N} |y_i|^p\right]^{\frac{1}{p}} \left[\sum_{i=1}^{N} \left(|x_i + y_i|^{(p-1)}\right)^{p'}\right]^{\frac{1}{p'}}.$$

Notice that $\left[\sum_{i=1}^{N}\left(|x_i+y_i|^{(p-1)}\right)^{p'}\right]^{\frac{1}{p'}}$ is a common factor of both terms on the right-hand-side. Moreover, $\frac{1}{p}+\frac{1}{p'}=1$ can be reorganized as $p=p'(p-1)$, meaning:

$$\left[\sum_{i=1}^{N}\left(|x_i+y_i|^{(p-1)}\right)^{p'}\right]^{\frac{1}{p'}}=\left[\sum_{i=1}^{N}|x_i+y_i|^{p}\right]^{\frac{1}{p'}}.$$

In summary, we have so far shown that:

$$\sum_{i=1}^{N}|x_i+y_i|^p\leq\left[\sum_{i=1}^{N}|x_i+y_i|^p\right]^{\frac{1}{p'}}\left(\|\{x_i\}\|_{\ell_p}+\|\{y_i\}\|_{\ell_p}\right).$$

Divide both sides by $\left[\sum_{i=1}^{N}|x_i+y_i|^p\right]^{\frac{1}{p'}}$ and use the fact that $1-\frac{1}{p'}=\frac{1}{p}$ to obtain Minkowski's Inequality:

$$\frac{\left[\sum_{i=1}^{N}|x_i+y_i|^p\right]^{1}}{\left[\sum_{i=1}^{N}|x_i+y_i|^p\right]^{\frac{1}{p'}}}=\left[\sum_{i=1}^{N}|x_i+y_i|^p\right]^{\frac{1}{p}}=\|\{x_i+y_i\}\|_{\ell_p}\leq\|\{x_i\}\|_{\ell_p}+\|\{y_i\}\|_{\ell_p}.$$

□

Metric Space Exercise 2e. Carefully conclude that $\left(R^N,d_p\right)$ is a metric space when $1<p<\infty$.

Proof. LTGBG, and let $\{x_i\}$, $\{y_i\}$, and $\{z_i\}$ be sequences in $\mathbb{R}^N$. To show that $(\mathbb{R}^N,d_p)$ is a metric space, we must verify the three properties in the definition of a metric.

(1) Prove that $d_p(\{x_i\},\{y_i\})\geq 0$ with equality if and only if $\{x_i\}=\{y_i\}$.

By definition, we have:

$$d_p(\{x_i\},\{y_i\})=\left(\sum_{i=1}^{N}|x_i-y_i|^p\right)^{\frac{1}{p}}.$$

Due to the absolute value(s), this quantity is clearly nonnegative for every $\{x_i\}$ and $\{y_i\}$ in $\mathbb{R}^N$. In particular, the sum evaluates to zero if and only if $x_i-y_i=0$ for all $1\leq i\leq N$, which occurs if and only if $\{x_i\}=\{y_i\}$.

(2) Prove d_p is symmetric, i.e. show that $d_p(\{x_i\},\{y_i\})=d_p(\{x_i\},\{y_i\})$.

Recall that the absolute value is a metric on $\mathbb{R}$, so it is symmetric. With this observation, we immediately obtain that d_p is symmetric via:

$$d_p(\{x_i\},\{y_i\})=\left(\sum_{i=1}^{N}|x_i-y_i|^p\right)^{\frac{1}{p}}=\left(\sum_{i=1}^{N}|y_i-x_i|^p\right)^{\frac{1}{p}}=d_p(\{y_i\},\{x_i\}).$$

(3) Prove d_p satisfies the triangle inequality, i.e. $d_p(\{x_i\},\{y_i\}) \le d_p(\{x_i\},\{z_i\}) + d_p(\{z_i\},\{y_i\})$.

From the definition of $d_p(\{x_i\},\{y_i\})$, we subtract and add $\{z_i\}$:

$$d_p(\{x_i\},\{y_i\}) = \left(\sum_{i=1}^{N} |x_i - y_i|^p\right)^{\frac{1}{p}} = \left(\sum_{i=1}^{N} |x_i - z_i + z_i - y_i|^p\right)^{\frac{1}{p}}.$$

Now we apply Minkowski's inequality to the right-most expression:

$$d_p(\{x_i\},\{y_i\}) = \left(\sum_{i=1}^{N} |x_i - z_i + z_i - y_i|^p\right)^{\frac{1}{p}} \le \left(\sum_{i=1}^{N} |x_i - z_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{N} |z_i - y_i|^p\right)^{\frac{1}{p}}.$$

In short, the preceding says $d_p(\{x_i\},\{y_i\}) \le d_p(\{x_i\},\{z_i\}) + d_p(\{z_i\},\{y_i\})$ as required.

We have shown that d_p has the three requisite properties to be a metric on $\mathbb{R}^N$, so $(\mathbb{R}^N, d_p)$ is indeed a metric space.

□