For the first homework, we are asked to complete a partially given proof. You may work in groups.

Metric Space Exercise 1.

Theorem 1. *Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of continuous functions from the interval $[a,b]$ into $\mathbb{R}$ that convergences pointwise on $[a,b]$ to the continuous function $f\colon [a,b]\to\mathbb{R}$. Also let $\{f_n\}_{n\in\mathbb{N}}$ be (pointwise) nonincreasing, i.e.,*

$$f_n(x) \geq f_{n+1}(x) \ , \text{ for each } x\in[a,b] \text{ and } n\in\mathbb{N}. \tag{D}$$

Show that $\{f_n\}_{n\in\mathbb{N}}$ converges uniformly to f on $[a,b]$ (by completing the below proof).

How does Dini's Theorem (stated below) follow from this exercise?

Theorem 2. *(Dini's Thm.) If $\{f_n\}_{n\in\mathbb{N}}$ be a montone sequence of $\mathbb{R}$-valued continuous functions on $[a,b]$ that convergences pointwise to the continuous function f on $[a,b]$, then $\{f_n\}_{n\in\mathbb{N}}$ converges uniformly to f on $[a,b]$.* ⟨*Here, montone means that* (D) *holds or* (I) *holds, where* (I) *is:*⟩

$$f_n(x) \leq f_{n+1}(x) \ , \text{ for each } x\in[a,b] \text{ and } n\in\mathbb{N}. \tag{I}$$

Proof. LTGBG. ⟨First, reduce to easier problem. WLOG, $f = 0 1_{[a,b]}$ for if f is not identically zero on $[a,b]$, then replace each f_n with $f_n - f$. In case you do do see this, the rest of this paragraph is the details. We will continue the proof using all the detail.⟩ For each $n\in\mathbb{N}$, define $g_n\colon\ [a,b]\to\mathbb{R}$ pointwise by

$$g_n := f_n - f.$$

Since $f_n \to f$ pointwise on $[a,b]$, we have that $g_n \to 0 1_{[a,b]}$ pointwise on $[a,b]$. Thus, by (D),

$$g_n(x) \geq g_{n+1}(x) \geq 0 \ , \text{ for each } x\in[a,b] \text{ and } n\in\mathbb{N}. \tag{D$_g$}$$

Note $f_n \rightrightarrows f$ on $[a,b]$ is equivalent to $g_n \rightrightarrows 0 1_{[a,b]}$ on $[a,b]$. We shall show the later by contradiction.

For each $n\in\mathbb{N}$, the nonnegative continuous function g_n must obtain it's supremum on the compact set $[a,b]$ and so there exists $x_n\in[a,b]$ such that

$$\sup_{x\in[a,b]} |g_n(x)| = \sup_{x\in[a,b]} g_n(x) = \max_{x\in[a,b]} g_n(x) = g_n(x_n).$$

The sequence $\{g_n(x_n)\}_{n\in\mathbb{N}}$ is a nonincreasing sequence from $[0,\infty)$ since

$$g_n(x_n) = \sup_{x\in[a,b]} g_n(x) \overset{\text{by } (\mathrm{D}_g)}{\geq} \sup_{x\in[a,b]} g_{n+1}(x) = g_{n+1}(x_{n+1}).$$

Towards a contradiction, assume that $\{g_n\}_{n\in\mathbb{N}}$ does not converge uniformly on $[a,b]$ to $0 1_{[a,b]}$. Thus the sequence $\{g_n(x_n)\}_{n\in\mathbb{N}}$ from $[0,\infty)$ does not converge to 0. Since the sequence $\{g_n(x_n)\}_{n\in\mathbb{N}}$

is nonincreasing, there is an $\varepsilon_0 > 0$ such that

$$g_n(x_n) > \varepsilon_0 \ , \text{ for each } n \in \mathbb{N}. \tag{1}$$

By the Bolzano-Weierstass Theorem, there is a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ and $x_0 \in [a,b]$ such that $x_{n_k} \longrightarrow x_0$ as $k \longrightarrow \infty$. Omit this green box – it was never used & the idea is wrong. You want

Without loss of generality, let $j, k \in \mathbb{N}$ such that $j < k$. Then $\boxed{\text{there exists } \varepsilon_{00} > 0 \text{ such that for}}$ a nice typo! every $n_j, n_k \in \mathbb{N}$

$$\begin{aligned} & g_{n_j}(x_{n_k}) \overset{\text{by } (D_g)}{\geq} g_{n_k}(x_{n_k}) \overset{\text{by } (1)}{>} \varepsilon_0 \\ \Rightarrow \quad & g_{n_j}(x_{n_k}) > \varepsilon_0 \\ \Rightarrow \quad & \lim_{k\to\infty} g_{n_j}(x_{n_k}) \geq \varepsilon_0 \\ \Rightarrow \quad & g_{n_j}(x_0) \geq \varepsilon_0 \end{aligned}$$

here the ε_0 & not some ε_{00} that depends on the j.

recall, "can lose" strict inequality in a limit.

b/c of here

Since $\lim_{j\to\infty} g_{n_j}(x_0) \geq \varepsilon_0$ we have that $g_{n_j} \not\to 01_{[a,b]}$ pointwise on $[a,b]$, a contradiction. Therefore $\{g_n\}_{n\in\mathbb{N}}$ converges uniformly on $[a,b]$ to $01_{[a,b]}$. Thus $\{f_n\}_{n\in\mathbb{N}}$ converges uniformly to f on the interval $[a,b]$. nice overall! □

How does *Dini's Theorem* follow from this exercise?

Let $\{h_n\}_{n\in\mathbb{N}}$ be a sequence of continuous functions that satisfy the properties of *Theorem 1*. Define $f_n := -h_n$ and $f := -h$. We can see that f_n and f are functions that satisfy the LTGBG conditions of *Theorem 2* (*Dini's Theorem*). Since $\{h_n\}_{n\in\mathbb{N}}$ converges uniformly to h on $[a,b]$ we have that $\{f_n\}_{n\in\mathbb{N}}$ converges uniformly to f on the interval $[a,b]$.