Exercise. Let $0 < a < 1$. Using the Residue Theorem, calculate the Cauchy principal value of
$$\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x}\,dx \,.$$

Solution. Observe that the function $e^{ax}/(1+e^x)$ has an infinite number of singularities in both the upper and lower half-planes; these occur at the points

$$z = (2n+1)\pi i \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

Hence if we employ expanding semicircles, the contribution due to the residues will result in an infinite series, which is undesirable. Moreover, there is no obvious way to estimate the contribution due to the semicircles themselves!

A better "return path" is revealed through careful examination of the integrand. The denominator of the function $e^{ax}/(1+e^x)$ is unchanged if z is shifted by $2\pi i$, whereas the numerator changes by a factor of $e^{2\pi ai}$. Thus if we consider the rectangular contour Γ_ρ in Fig. 6.6, the contribution from γ_3 is easy to assess; it's merely $-e^{2\pi ai}$ times the contribution from γ_1 (negative because the path runs from right to left). Therefore

$$\int_{\gamma_3} \frac{e^{az}}{1+e^z}\,dz = -e^{2\pi ai} \int_{\gamma_1} \frac{e^{az}}{1+e^z}\,dz.$$

For $\gamma_2 : z = \rho + it, 0 \le t \le 2\pi$, we have

$$\left| \int_{\gamma_2} \frac{e^{az}}{1+e^z}\,dz \right| = \left| \int_0^{2\pi} \frac{e^{a(\rho+it)}}{1+e^{\rho+it}}\, i\,dt \right| \le \frac{e^{a\rho}}{e^\rho - 1} \cdot 2\pi,$$

which goes to zero as $\rho \to \infty$ since $a < 1$.

Similarly, on $\gamma_4 : z = -\rho + i(2\pi - t), 0 \le t \le 2\pi$, we have

$$\left| \int_{\gamma_4} \frac{e^{az}}{1+e^z}\,dz \right| = \left| \int_0^{2\pi} \frac{e^{a[-\rho+i(2\pi-t)]}}{1+e^{-\rho+i(2\pi-t)}}\,(-i)\,dt \right| \le \frac{e^{-a\rho}}{1-e^{-\rho}} \cdot 2\pi,$$

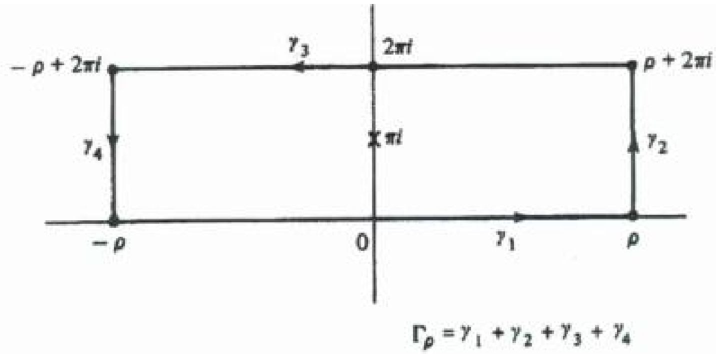

Figure 6.6 Closed contour for Example 3.

again approaching zero as $\rho \to \infty$ since $a > 0$.

As a result, on taking the limit as $\rho \to \infty$ we have

$$\lim_{\rho \to \infty} \int_{\Gamma_\rho} \frac{e^{az}}{1+e^z}\, dz = \left(1 - e^{a2\pi i}\right) \text{ p.v. } \int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x}\, dx. \tag{8}$$

Now we use residue theory to evaluate the contour integral in Eq. (8). For each $\rho > 0$, the function $e^{az}/(1+e^z)$ is analytic inside and on Γ_ρ except for a simple pole at $z = \pi i$, the residue there being given by

$$\text{Res}(\pi i) = \left. \frac{e^{az}}{\dfrac{d}{dz}(1+e^z)} \right|_{z=\pi i} = \frac{e^{a\pi i}}{e^{\pi i}} = -e^{a\pi i} \tag{9}$$

(recall Example 2, Sec. 6.1). Consequently, putting Eqs. (8) and (9) together we obtain

$$\begin{aligned} \text{p.v.} \int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x}\, dx &= \frac{1}{1-e^{a2\pi i}} \cdot (2\pi i)\left(-e^{a\pi i}\right) \\ &= \frac{-2\pi i}{e^{-a\pi i} - e^{a\pi i}} \\ &= \frac{\pi}{\sin a\pi}. \quad \blacksquare \end{aligned}$$

The above mentioned *Example 2, Sec. 6.1* is the following *Useful Fact* shown in class.

Theorem 2. *Let two functions p and q be analytic at a point z_0. If*

$$p(z_0) \neq 0, \quad q(z_0) = 0, \quad \text{and} \quad q'(z_0) \neq 0,$$

then z_0 is a simple pole of the quotient $p(z)/q(z)$ and

$$\operatorname*{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}. \tag{2}$$