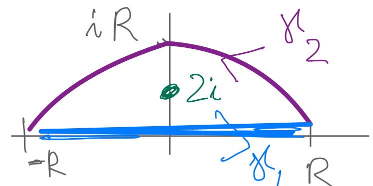
Exercise. Using the Residue Theorem, calculate
$$\int_0^\infty \frac{\cos 3x}{x^2+4}\,dx\ .$$



$\Gamma_R = \gamma_1 \cup \gamma_2$

Proof. Consider,
$$f(z) = \frac{e^{3iz}}{z^2+4}.$$
Clearly, $f(z) \in H(\mathbb{C} \setminus \{-2i, 2i\})$. Let $R \in \mathbb{R}$ such that $R > \cancel{1}$. Define $\gamma_1 : [-R, R]$ such that $\gamma_1(t){:=}t$ and $\gamma_2 : [0,\pi]$ such that $\gamma_2(t){:=}Re^{it}$. We denote Γ_R as the join of γ_1 and γ_2. Therefore,

want $R>2$ so that the point $(2i)$ is "inside" Γ_R.

$$\int_{\Gamma_R} f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz. \tag{1}$$
We apply the Residue Theorem over the closed path Γ_R,
$$\int_{\Gamma_R} f(z)\,dz = 2\pi i\,[\mathrm{Res}(f(z), 2i)]\,\mathrm{Ind}_{\Gamma_R}(2i).$$
By construction of Γ_R, $\mathrm{Ind}_{\Gamma_R}(2i) = 1$. Now, we have $e^{3iz}, z^2+4 \in H(\mathbb{C})$ with $(2i)^2+4=0$ such that $e^{3i(2i)} \neq 0$ and $(z^2+4)' = 2z$ evaluated at $2i$ is nonzero. We compute,
$$\int_{\Gamma_R} f(z)\,dz = 2\pi i\,[\mathrm{Res}\,(f(z),2i)] = 2\pi i \frac{e^{3i(2i)}}{2(2i)} = \frac{\pi}{2e^6}. \tag{2}$$
Applying the Maximum Modulus Principle, since e^{3iz} is entire, we have that $|e^{3iz}|$ obtains its max on the boundary of Γ_R^*. Letting $z = x+iy$, we observe that,
$$\left|e^{3iz}\right| = \left|e^{-3y+3xi}\right| = e^{-3y}.$$
Therefore,
$$\max_{z\in\gamma_2^*}\left|e^{3iz}\right| \le \max_{z\in\Gamma_R^*}\left|e^{3iz}\right| = \max_{z\in\Gamma_R^*} e^{-3y} = e^0 = 1.$$
From the reverse triangle inequality $|z^2+4| \ge |z|^2 - 4$. Using the ML lemma, we observe that,
$$\begin{aligned}\lim_{R\to\infty}\left|\int_{\gamma_2} f(z)\,dz\right| &= \lim_{R\to\infty} l(\gamma_2)\left[\max_{z\in\gamma_2^*}\left|\frac{e^{3iz}}{z^2+4}\right|\right]\\ &\le \lim_{R\to\infty} \pi R\left[\max_{z\in\gamma_2^*}\left|\frac{1}{|z|^2-4}\right|\right]\\ &= \lim_{R\to\infty}\frac{\pi R}{R^2-4}\\ &= 0.\end{aligned}$$
Which implies,
$$\lim_{R\to\infty}\int_{\gamma_2} f(z)\,dz = 0 \tag{3}$$

Substituting (2) and (3) into (1) and taking the limit as $R \to \infty$,

$$\begin{aligned}\frac{\pi}{2e^6} &= \lim_{R\to\infty}\int_{\gamma_1} f(z)\,dz \\ &= \lim_{R\to\infty}\int_{-R}^{R} f(t)\,dt \\ &= \int_{-\infty}^{\infty}\frac{e^{3it}}{t^2+4}\,dt \\ &= \int_{-\infty}^{\infty}\frac{\cos 3t}{t^2+4}\,dt + i\int_{-\infty}^{\infty}\frac{\sin 3t}{t^2+4}\,dt.\end{aligned}$$

Hence,

$$\int_{-\infty}^{\infty}\frac{\cos 3t}{t^2+4}\,dt = \frac{\pi}{2e^6}.$$

Since $\frac{\cos 3t}{t^2+4}$ is an even function, we conclude,

$$\boxed{\int_{0}^{\infty}\frac{\cos 3t}{t^2+4}\,dt = \frac{\pi}{4e^6}.}$$

very nice.

□

A common mistake: Using Residue Thm w/ $\frac{\cos 3z}{z^2+4}$ instead of $\frac{e^{i3z}}{z^2+4}$.

Recall $\cos z := \frac{e^{iz}+e^{-iz}}{2}$ for $z \in \mathbb{C}$. So if $\theta \in \mathbb{R}$, then

$\cos(i\theta) = \frac{e^{i(i\theta)}+e^{-i(i\theta)}}{2} = \frac{e^{\theta}+e^{-\theta}}{2} \xrightarrow[\theta\in\mathbb{R}]{\theta\to\infty} \infty.$

So when using ML w/ $\int_{\gamma_2}$ get; when $R>2$ and $\gamma_2(t) := Re^{it}$, for $0\le t\le \pi$

$\max_{z\in\gamma_2^*}\left|\frac{\cos 3z}{z^2+4}\right| \underset{z=iR\in\gamma_2^*}{\overset{\text{take}}{\geq}} \left|\frac{\cos(i3R)}{(iR)^2+4}\right| \overset{R>2}{=} \frac{e^{3R}+e^{-3R}}{2(R^2-4)} \xrightarrow{R\to\infty} \infty.$