Exercise. Let $R > 0$ and $z_0 \in \mathbb{C}$.

(a) Let $f \in H\left(B_R\left(z_0\right)\right)$ have a zero of order m at z_0. Show $\dfrac{1}{f}$ has a pole of order m at z_0.

(b) Let $f \in H\left(B'_R\left(z_0\right)\right)$ have a pole of order m at z_0. Show $\dfrac{1}{f}$ has a removable singularity at z_0, and furthermore, if we extend $\frac{1}{f}$ to $\widetilde{\frac{1}{f}}$ by defining $\widetilde{\frac{1}{f}}\left(z_0\right) = 0$, then $\widetilde{\frac{1}{f}}$ is holomorhic at z_0 and has a zero of order m at z_0.

(c) What is the order of the pole of

$$h\left(z\right) := \frac{1}{\left(2\cos z - 2 + z^2\right)^2}$$

at $z = 0$? Explain your answer.

Remark. Loosely speaking, this exercise shows that, at $z = z_0$,

(a) f has zero of order $m \Rightarrow \frac{1}{f}$ has pole of order m

(b) f has pole of order $m \Rightarrow \frac{1}{f}$ has zero of order m.

(a) Their Thm 16 = our script's Ch 3's Thm 1.1 (p 27). Their Lemma 7 is our *Singularities of Holomorphic Fuctions*'s TFAE (2b).

Suppose that f has a zero of order m at z_0. Then by Theorem 16 there is a function $g(z)$ analytic at z_0 with $g(z_0) \neq 0$ and $f(z) = (z - z_0)^m g(z)$. Consequently $\dfrac{1}{f(z)} = \dfrac{1/g(z)}{(z - z_0)^m}$ and $1/f$ has a pole of order m at z_0 by Lemma 7.

(b) Their Lemma 7 is our *Singularities of Holomorphic Fuctions*'s TFAE (2b).

Conversely, if f has a pole of order m at z_0 it follows from the properties of g given in Lemma 7 that $\dfrac{1}{f(z)} = \dfrac{(z - z_0)^m}{g(z)}$ has a removable singularity at z_0. Defining $1/f(z_0) = 0$ results in $1/f$ having a zero of order m at z_0.

(c)

8 (since $1/f(z)$ has a zero of order 8 at $z = 0$)