Exercise.
Let $f, g \in H(G)$, with G is open and connected, satisfy

$$f(z)\,g(z) = 0 \quad \text{for each} \quad z \in G\ .$$

Show that $f(z) = 0$ for each $z \in G$ or $g(z) = 0$ for each $z \in G$.

Proof. Let $f, g \in H(G)$, with G is open and connected, satisfy

$$f(z)\,g(z) = 0 \quad \text{for each} \quad z \in G\ . \tag{1}$$

Denote the zero set of f (resp. g) by Z_f (resp. Z_g), i.e.,

$$Z_f := \{z \in G \colon f(z) = 0\}$$

$$Z_g := \{z \in G \colon g(z) = 0\}\ .$$

We want to show that $Z_f = G$ or $Z_g = G$. $\langle$The theorem on zeros of holomorphic functions (Thm III.1.1) tells us either $Z_f = G$ or $Z_f = I_{Z_f}$, where I_A denotes the isolated point of a set $A \subset \mathbb{C}$. Similarly for g.$\rangle$

Way 1. Assume that $Z_g \neq G$. Then there exists $z_0 \in G$ such that $g(z_0) \neq 0$, and so, by the continuity of g, there also exists $\delta > 0$ such that $g(z) \neq 0$ for each $z \in B_\delta(z_0)$. By (1), $f(z) = 0$ for each $z \in B_r(z_0)$. Thus z_0 is a non-isolated zero of f , and so, by the theorem on zeros of holomorphic functions (Thm III.1.1), $Z_f = G$.

Way 2. Assume that $Z_f \neq G$ and $Z_g \neq G$. Then by the theorem on zeros of holomorphic functions (Thm III.1.1), Z_f and Z_g are each (at most) countable. Note that $Z_{fg} = Z_f \cup Z_g$ and that (1) gives $Z_{fg} = G$. Thus G is (at most) countable. A contradiction. Thus $Z_f = G$ or $Z_g = G$. $\square$