Exercise.
Let $f \in H(B_1(0))$ satisfy that
(i) $|f(z)| \leq 1$ for each $z \in B_1(0)$
(ii) $f(0) = 0$.
Show that
(a) $|f(z)| \leq |z|$ for each $z \in B_1(0)$,
(b) $|f'(0)| \leq 1$.
If, furthermore, $|f(z_0)| = |z_0|$ for some $z_0 \in B_1'(0)$, show that
(c) there exists $c \in \mathbb{C}$ with $|c| = 1$ such that $f(z) = cz$ for each $z \in B_1(0)$.
Recall: $B_1(0) := \{z \in \mathbb{C} \colon |z| < 1\}$.
Remark: this exercise is known as Schwarz's Lemma.

Proof. Let $f \in H(B_1(0))$ satisfy: (i) $|f(z)| \leq 1$ for each $z \in B_1(0)$ and (ii) $f(0) = 0$.

Define $g \colon B_1(0) \to \mathbb{C}$ by

$$g(z) := \begin{cases} \frac{f(z)}{z} & \text{, if } z \in B_1'(0) \\ f'(0) & \text{, if } z = 0. \end{cases}$$

Note $g \in H(B_1'(0))$ since $f \in H(B_1'(0))$. Also $g \in C(B_1(0))$ since $f \in H(B_1(0))$ and

$$\lim_{z\to 0} g(z) \overset{\text{by def of } g}{=} \lim_{z\to 0} \frac{f(z)}{z} \overset{\text{by (ii)}}{=} \lim_{z\to 0} \frac{f(z) - f(0)}{z - 0} \underset{f\in H(B_1(0))}{\overset{\text{since}}{=}} f'(0).$$

Thus $g \in H(B_1(0))$ since $g \in H(B_1'(0)) \cap C(B_1(0))$ [cf., eg, Morea's Thm with Cauchy's Thm for triangles].

For $0 < r < 1$, applying the Max. Modulus Principle (Cor. III.1.6) to $g \in H(B_r(0)) \cap C\left(\overline{B_r(0)}\right)$,

$$\max_{z\in\overline{B_r(0)}} |g(z)| \overset{\text{Max Mod}}{\underset{\text{Princ}}{=}} \max_{z\in\partial B_r(0)} |g(z)| = \max_{z\in\partial B_r(0)} \left|\frac{f(z)}{z}\right| = \max_{z\in\partial B_r(0)} \frac{|f(z)|}{r} \underset{\text{(i)}}{\overset{\text{by}}{\leq}} \frac{1}{r}. \tag{1}$$

Letting $r \nearrow 1$ in (1) gives

$$\max_{z\in B_1(0)} |g(z)| \leq 1. \tag{2}$$

(a) Note $|f(z)| \leq |z|$ for each $z \in B_1'(0)$ by the definition of g and the inequality (2). Clearly $|f(z)| \leq |z|$ when $z = 0$ since $f(0) \overset{\text{by (ii)}}{=} 0$.

(b) Note (b) holds since $|f'(0)| \overset{\text{def of } g}{=} |g(0)| \overset{\text{by (2)}}{\leq} 1$.

(c) Now assume furthermore $z_0 \in B_1'(0)$ satisfies $|f(z_0)| = |z_0|$. Then $|g(z_0)| = 1 \underset{(2)}{\overset{\text{by}}{=}} \max_{z\in B_1(0)} |g(z)|$.

So the max of the modulus of $g \in H(B_1(0))$ is attained at $z_0 \in B_1(0)$. By the Max Modulus Principle, g is constant on $B_1(0)$ so there is $c \in \mathbb{C}$ such that

$$g(z) = c \text{ for each } z \in B_1(0). \tag{3}$$

Thus

$$f(z) = cz$$

for each $z \in B_1'(0)$ by the definition of g as well as for $z = 0$ since $f(0) \overset{\text{by (ii)}}{=} 0$.

Note $|c| = 1$ since $|g(z_0)| = 1$. □