Exercise. Problem Source: Quals 1995.
Let $f \in H(\mathbb{C})$ satifsy, for some constants $A, B \in \mathbb{R}$ and $k \in \mathbb{N}$,

$$|f(z)| \ \le \ A\,|z|^k + B \tag{1}$$

for each $z \in \mathbb{C}$. Prove that f is a polynomial.
Hint: use the CIF (see Cor. II.2.24a from class).

Proof. LTGBG. Since f is entire

$$f\,(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}\, z^n$$

for all $z \in \mathbb{C}$.

Fix $n \in \mathbb{N}$ such that $n > k$ Let $R > 0$ and define $\gamma_R\colon [0, 2\pi] \to \mathbb{C}$ by $\gamma\,(t) = Re^{it}$. From Cauchy integral formula (Cor. II.2.24)

$$f^{(n)}\,(0) \ = \ \frac{n!}{2\pi i}\int_{\gamma_R} \frac{f\,(z)}{z^{n+1}}\,dz\ .$$

Thus

$$\begin{aligned}
\left|f^{(n)}\,(0)\right| \ &= \ \frac{n!}{2\pi}\left|\int_{\gamma_R} \frac{f\,(z)}{z^{n+1}}\,dz\right| \\
&\overset{\text{(ML)}}{\le} \ \frac{n!}{2\pi}\left[\sup_{z\in\gamma_R^*}\left|\frac{f\,(z)}{z^{n+1}}\right|\right] l\,(\gamma_R^*) \\
&\overset{\text{by (1)}}{\le} \ \frac{n!}{2\pi}\left[\frac{AR^k + B}{R^{n+1}}\right](2\pi R) \\
&= \ n!\left[\frac{A}{R^{n-k}} + \frac{B}{R^n}\right] \xrightarrow{R\to\infty\text{ , since } n>k} 0\ .
\end{aligned}$$

Thus, if $n > k$, then $f^{(n)}\,(0) = 0$ and so

$$f\,(z) = \sum_{n=0}^{k} \frac{f^{(n)}(0)}{n!}\, z^n$$

for all $z \in \mathbb{C}$. $\square$