Thm. 2.14. Cauchy's Integral Formula for starlike sets. Let:

1. $\gamma^* \subset G \subset \mathbb{C}$
2. $f \in H(G)$
3. $a \in \mathbb{C} \setminus \gamma^*$

where γ is a closed contour and G is open and starlike. Then

$$[f(a)] \cdot [\mathrm{Ind}_\gamma(a)] \;=\; \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-a}\, dz \;. \tag{CIF}$$

Exercise. Let $\alpha \in \mathbb{C} \setminus \{0\}$ with $|\alpha| \neq 1$. Let $\gamma\colon [0, 2\pi] \to \mathbb{C}$ be given by $\gamma(\theta) := e^{i\theta}$.
(a) Using Cauchy's Integral Formula (CIF) for starlike sets, calculate

$$\int_\gamma \frac{dz}{(z-\alpha)\left(z - \frac{1}{\alpha}\right)}$$

for the case that $|\alpha| < 1$ as well as the case that $|\alpha| > 1$.
When applying CIF for starlike sets, following the notation in the Script of this theorem [Thm. II.2.14, p. 21], clearly indicate what is your: open starlike set G, function $f\colon G \to \mathbb{C}$, and $z_0 \in G \setminus \gamma^*$.
(b) Using part (a), calculate

$$\int_0^{2\pi} \frac{d\theta}{1 - 2\alpha\cos\theta + \alpha^2}$$

for the case that $|\alpha| < 1$ as well as the case that $|\alpha| > 1$. Hint. $e^{i\theta} + e^{-i\theta} = 2\cos\theta$.

Solution Box. Put your final solution in the boxes provided below and then show your justification your solutions below this Solution Box.
(a)

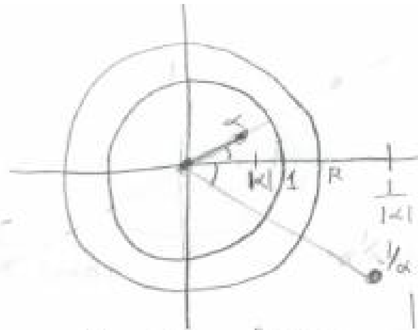
$$\int_\gamma \frac{dz}{(z-\alpha)\left(z-\frac{1}{\alpha}\right)} = \begin{cases} \boxed{\dfrac{2\pi\, i}{\alpha - \frac{1}{\alpha}} \overset{\text{i.e.}}{=} 2\pi\, i \left(\dfrac{\alpha}{\alpha^2 - 1}\right)} & \text{if } |\alpha| < 1 \\[2ex] \boxed{\dfrac{2\pi\, i}{\frac{1}{\alpha} - \alpha} \overset{\text{i.e.}}{=} 2\pi\, i \left(\dfrac{\alpha}{1 - \alpha^2}\right)} & \text{if } |\alpha| > 1 \;. \end{cases}$$

(b)

$$\int_0^{2\pi} \frac{d\theta}{1 - 2\alpha\cos\theta + \alpha^2} = \begin{cases} \boxed{\dfrac{2\pi}{1-\alpha^2}} & \text{if } |\alpha| < 1 \\[2ex] \boxed{\dfrac{2\pi}{\alpha^2 - 1}} & \text{if } |\alpha| > 1 \;. \end{cases}$$

Justification/Proof of the Solutions:

Solution to (a) LTGBG.



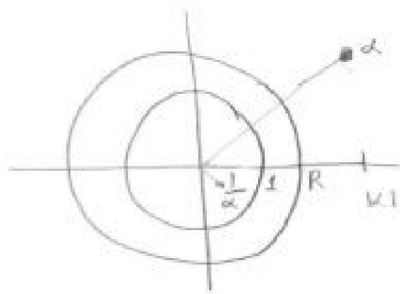
Case $|\alpha| < 1$.

- Pick R st $1 < R < \frac{1}{|\alpha|}$.
- Let $G = B_R(0)$ and consider $f : G \to \mathbb{C}$ where $f(z) = \frac{1}{z - \frac{1}{\alpha}}$.
- So $f \in H(G)$ and $\alpha \in G \setminus \gamma^*$.
- Compute

$$\int_\gamma \frac{dz}{(z-\alpha)(z-\frac{1}{\alpha})} = \int_\gamma \frac{f(z)}{z-\alpha}\,dz \overset{\text{CIF}}{=} (2\pi i)(\mathrm{Ind}_\gamma \alpha)(f(\alpha))$$

$$= (2\pi i)(1)\left(\frac{1}{\alpha - \frac{1}{\alpha}}\right).$$

Case $|\alpha| > 1$



- Pick R s.t. $1 < R < |\alpha|$.
- Let $G = B_R(0)$ and consider $f : G \to \mathbb{C}$ where $f(z) = \frac{1}{z-\alpha}$.
- So $f \in H(G)$ and $\frac{1}{\alpha} \in G \setminus \gamma^*$.
- Compute

$$\int_\gamma \frac{dz}{(z-\alpha)(z-\frac{1}{\alpha})} = \int_\gamma \frac{f(z)}{z-\frac{1}{\alpha}}\,dz \overset{\text{CIF}}{=} 2\pi i \left(\mathrm{Ind}_\gamma\left(\tfrac{1}{\alpha}\right)\right)\left(f\left(\tfrac{1}{\alpha}\right)\right)$$

$$= (2\pi i)(1)\left(\frac{1}{\frac{1}{\alpha} - \alpha}\right)$$

Solution to (b)

$$\int_0^{2\pi} \frac{d\theta}{1-2\alpha\cos\theta+\alpha^2} = \int_0^{2\pi} \frac{d\theta}{1-2\alpha(\frac{e^{i\theta}+e^{-i\theta}}{2})+\alpha^2} = \int_0^{2\pi} \frac{d\theta}{1-\alpha(e^{i\theta}+e^{-i\theta})+\alpha^2}$$

$$= -i\int_0^{2\pi} \frac{ie^{i\theta}d\theta}{e^{i\theta}-\alpha(e^{i2\theta}+1)+\alpha^2 e^{i\theta}}$$

$$= -i\int_\gamma \frac{dz}{z-\alpha(z^2+1)+\alpha^2 z} = +\frac{i}{\alpha}\int_0^{2\pi} \frac{dz}{-\frac{z}{\alpha}+z^2+1-\alpha z}$$

$$= \frac{i}{\alpha}\int_\gamma \frac{dz}{z^2-(\alpha+\frac{1}{\alpha})z+1} = \frac{i}{\alpha}\int_\gamma \frac{dz}{(z-\alpha)(z-\frac{1}{\alpha})},$$

$$= \frac{-1}{i\alpha}\int_\gamma \frac{dz}{(z-\alpha)(z-\frac{1}{\alpha})}$$

$$\overset{(a)}{=} \begin{cases} \frac{-1}{i\alpha}\cdot\frac{2\pi i\alpha}{\alpha^2-1} = \frac{2\pi}{1-\alpha^2} & \text{if } |\alpha|<1 \\ \frac{-1}{i\alpha}\ \frac{2\pi i\alpha}{1-\alpha^2} = \frac{2\pi}{\alpha^2-1} & \text{if } |\alpha|>1 \end{cases}$$

Remark You can do (a) without the CIF as follows.

① Do a PFD (= partial fraction decomposition), over $\mathbb{C}$ (not $\mathbb{R}$)

$$\frac{1}{(z-\alpha)(z-\frac{1}{\alpha})} = \frac{A}{z-\alpha}+\frac{B}{z-\frac{1}{\alpha}} \overset{\text{work}}{=} A\left[\frac{1}{z-\alpha}+\frac{-1}{z-\frac{1}{\alpha}}\right]$$

where $A = \frac{\alpha}{\alpha^2-1}$.

② $$\int_\gamma \frac{dz}{(z-\alpha)(z-\frac{1}{\alpha})} \overset{①}{=} A\left[\int_\gamma \frac{dz}{z-\alpha} - \int_\gamma \frac{dz}{z-\frac{1}{\alpha}}\right]$$

$$\overset{\boxed{\text{def of winding number}}}{=} A\left[2\pi i\, \mathrm{Ind}_\gamma(\alpha) - 2\pi i\, \mathrm{Ind}_\gamma \tfrac{1}{\alpha}\right]$$

$$= \begin{cases} (2\pi i)A\ [1-0] & \text{if } |\alpha|<1 \\ (2\pi i)A\ [0-1] & \text{if } |\alpha|>1. \end{cases}$$