Exercise.

a) Find the partial fraction decomposition (PFD) over $\mathbb{C}$ of $\frac{1}{1+z^2}$.

Hint: $\frac{1}{1+z^2} = \frac{1}{(z-i)(z+i)} \overset{\text{PFD}}{=} \frac{A}{z-i} + \frac{B}{z+i}$ for some $A, B \in \mathbb{C}$. Find A and B.

b) Evaluate (without parametrizing the curve γ, but rather by using Cauchy's Integral Formula and the above PFD)

$$\int_\gamma \frac{dz}{1+z^2}$$

for the following $\gamma\colon [0, 2\pi] \to \mathbb{C}$.

1. $\gamma(t) := 1 + e^{it}$
2. $\gamma(t) := -i + e^{it}$
3. $\gamma(t) := 2e^{it}$
4. $\gamma(t) := 3i + 3e^{it}$

Useful Ideas

Thm. 2.14. Cauchy's Integral Formula for starlike sets. Let:

1. $\gamma^* \subset G \subset \mathbb{C}$
2. $f \in H(G)$
3. $a \in \mathbb{C} \setminus \gamma^*$

where γ is a closed contour and G is open and starlike. Then

$$[f(a)] \cdot [\mathrm{Ind}_\gamma(a)] = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-a}\, dz\ , \tag{CIF}$$

where the index of γ w.r.t. a (also called the winding number of γ around a) is defined by

$$\mathrm{Ind}_\gamma(a) := \frac{1}{2\pi i} \int_\gamma \frac{dz}{z-a}\ . \tag{Def. 2.13}$$

Example ⊙. Let $\gamma\colon [0, 2\pi n] \to \mathbb{C}$ be given by $\gamma(t) = a + re^{ict}$ where: $n \in \mathbb{N}$, $a \in \mathbb{C}$, $r > 0$, and $c \in \{\pm 1\}$. Then $\mathrm{Ind}_\gamma(a) = c\,n$.

Rmk. 2.19. $\mathbb{C} \setminus (\text{a closed contour})^*$ has exactly one unbounded (connected) component.

Thm. 2.20. Let γ be a closed contour. Then

1. $\mathrm{Ind}_\gamma\colon \mathbb{C} \setminus \gamma* \to \mathbb{Z}$ is constant on each (connenected) component of $\mathbb{C} \setminus \gamma*$
2. $\mathrm{Ind}_\gamma(a) = 0$ for each a in the unbounded of (connenected) component $\mathbb{C} \setminus \gamma*$.

Exercise.
a) Find the partial fraction decomposition (PFD) over $\mathbb{C}$ of $\frac{1}{1+z^2}$.
Hint: $\frac{1}{1+z^2} = \frac{1}{(z-i)(z+i)} \overset{\text{PFD}}{=} \frac{A}{z-i} + \frac{B}{z+i}$ for some $A, B \in \mathbb{C}$. Find A and B.
b) Evaluate (without parametrizing the curve γ, but rather by using Cauchy's Integral Formula and the above PFD)

$$\int_\gamma \frac{dz}{1+z^2}$$

for the following $\gamma\colon [0, 2\pi] \to \mathbb{C}$.
1. $\gamma(t) := 1 + e^{it}$
2. $\gamma(t) := -i + e^{it}$
3. $\gamma(t) := 2e^{it}$
4. $\gamma(t) := 3i + 3e^{it}$

(a)

$$A = \frac{1}{2i} = \frac{-i}{2} \qquad \text{and} \qquad B = \frac{-1}{2i} = \frac{i}{2}$$

(b) LTGBG. Define

$$I := \int_\gamma \frac{dz}{1+z^2} .$$

(i) First, do a PFD (= partial fraction decomposition) over $\mathbb{C}$ (not $\mathbb{R}$):

$$\frac{1}{1+z^2} = \frac{1}{(z-i)(z+i)} = \frac{A}{z-i} + \frac{B}{z+i} \overset{\text{work}}{=} \frac{1}{2i}\left(\frac{1}{z-i} - \frac{1}{z+i}\right).$$

(ii) So by CIF (or just def. of winding number),
as long as $i, -i \notin \gamma^*$,

$$\boxed{I :=} \int_\gamma \frac{dz}{1+z^2} = \pi\left[\frac{1}{2\pi i}\int_\gamma \frac{dz}{z-i} - \frac{1}{2\pi i}\int_\gamma \frac{dz}{z+i}\right]$$

$$= \pi\left[\operatorname{Ind}_\gamma(i) - \operatorname{Ind}_\gamma(-i)\right]$$

---

(1) $I = \pi\,[0-0] = 0$

(2) $I = \pi\,[0-1] = -\pi$

(3) $I = \pi\,[1-1] = 0$

(4) $I = \pi\,[1-0] = \pi$