**Exercise**. Compute

$$\int_0^{2\pi} e^{\cos t}\left[\cos\left(t+\sin t\right)\right]\,dt \qquad \text{and} \qquad \int_0^{2\pi} e^{\cos t}\left[\sin\left(t+\sin t\right)\right]\,dt$$

by computing $\int_\gamma e^z\,dz$ where $\gamma\colon [0,2\pi]\to\mathbb{C}$ is given by $\gamma(t) := e^{it}$.

By the FTC for path integral ⟨thm 2.8⟩,

$$\int_\gamma e^z\,dz = 0 \tag{1}$$

⟨since $\gamma$ is a closed piecewise smooth curve, the function $f(z) = e^z$ is entire, and $f'(z) = e^z$ is continuous on $\gamma^*$ we have $\int_\gamma f'(z)\,dz = 0$⟩. On the other hand, by definition of path integral,

$$\int_\gamma e^z dz = \int_0^{2\pi} e^{\gamma(t)}\left(\gamma'(t)\right)\,dt = \int_0^{2\pi} e^{e^{it}}\left(i\,e^{it}\right)\,dt\ . \tag{2}$$

So $\int_0^{2\pi} e^{e^{it}} e^{it}\,dt = 0$ by (1) and (2). Thus

$$\begin{aligned}
0 &= \int_0^{2\pi} e^{e^{it}} e^{it}\,dt \\
&= \int_0^{2\pi} e^{\cos t + i\sin t}\,e^{it}\,dt \\
&= \int_0^{2\pi} e^{\cos t}\,e^{i(t+\sin t)}\,dt \\
&= \int_0^{2\pi} e^{\cos t}\cdot\left[\cos\left(t+\sin t\right) + i\sin\left(t+\sin t\right)\right]\,dt \\
&= \int_0^{2\pi} e^{\cos t}\,\cos\left(t+\sin t\right)\;dt \;+\; i\int_0^{2\pi} e^{\cos t}\,\sin\left(t+\sin t\right)\,dt.
\end{aligned}$$

Note that

$$\int_0^{2\pi} e^{\cos t}\,\cos\left(t+\sin t\right)\;dt \;\in\mathbb{R}$$
$$\int_0^{2\pi} e^{\cos t}\,\sin\left(t+\sin t\right)\,dt. \;\in\mathbb{R}\ .$$

So

$$\int_0^{2\pi} e^{\cos t}\,\cos\left(t+\sin t\right)\;dt \;= 0$$

and

$$\int_0^{2\pi} e^{\cos t}\,\sin\left(t+\sin t\right)\,dt. \;= 0\ .$$