<u>The Summation by Parts Formula</u>. Let $\{a_n\}_{n=1}^N$ and $\{b_n\}_{n=1}^N$ be finite sequences of complex numbers. Then for $N > M > 1$

$$\sum_{n=M}^{N} a_n b_n \;=\; a_N B_N \;-\; a_M B_{M-1} \;-\; \sum_{n=M}^{N-1} \left(a_{n+1} - a_n\right) B_n \qquad \text{where} \qquad B_k = \sum_{l=1}^{k} b_l \; . \tag{1}$$

Hint for proof of summation by parts formula: substitute $b_n = B_n - B_{n-1}$ in the sum on the left. You may use, without proving, the *Summation by Parts Forumula.*

Exercise. Prove the power series $\displaystyle\sum_{n=1}^{\infty} \frac{z^n}{n}$ converges for each $z \in \mathbb{C}$ with $|z| = 1$ except $z = 1$.

Hint. Use the above summation by parts. Note if $z \in \mathbb{C} \setminus \{1\}$ with $|z| = 1$, then for each $k \in \mathbb{N}$

$$\left|\sum_{n=1}^{k} z^n\right| \;=\; \left|\frac{z - z^{k+1}}{1-z}\right| \;\leq\; \frac{|z| + \left|z^{k+1}\right|}{|1-z|} \;=\; \frac{2}{|1-z|} \; .$$

Proof. The power series $\sum_{n=1}^{\infty} \frac{z^n}{n}$ diverges when $z = 1$ since the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Now, fix $z \in \mathbb{C}$ with $|z| = 1$ but $z \neq 1$. Towards showing that the sequence $\left\{\sum_{n=1}^{N} \frac{z^n}{n}\right\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbb{C}$, let $\varepsilon > 0$. Pick $M_0 \in \mathbb{N}$ such that

$$\frac{1}{M_0} \cdot \frac{2}{|1-z|} \leq \frac{\varepsilon}{3} \; . \tag{2}$$

Fix $N, M \in \mathbb{N}$ such that $N > M \geq M_0$.

The summation by parts formula (1) ⟨with $a_n = \frac{1}{n}$ and $b_n = z^n$⟩ gives

$$\sum_{n=M}^{N} \frac{z^n}{n} = \frac{1}{N} \sum_{n=1}^{N} z^n - \frac{1}{M} \sum_{n=1}^{M-1} z^n - \sum_{n=M}^{N-1} \left(\frac{1}{n+1} - \frac{1}{n}\right) \cdot \left(\sum_{k=1}^{n} z^k\right) \; . \tag{3}$$

Note that for each $k \in \mathbb{N}$

$$\left|\sum_{n=1}^{k} z^n\right| \;=\; \left|\frac{z - z^{k+1}}{1-z}\right| \;\leq\; \frac{|z| + \left|z^{k+1}\right|}{|1-z|} \;=\; \frac{2}{|1-z|} \; . \tag{4}$$

Thus

$$\begin{aligned}
\left|\sum_{n=M}^{N} \frac{z^n}{n}\right| &\overset{\text{by (3)}}{\leq} \frac{1}{N} \left|\sum_{n=1}^{N} z^n\right| + \frac{1}{M} \left|\sum_{n=1}^{M-1} z^n\right| + \sum_{n=M}^{N-1} \left|\frac{1}{n+1} - \frac{1}{n}\right| \cdot \left|\sum_{k=1}^{n} z^k\right| \\
&\overset{\text{by (4)}}{\leq} \frac{1}{N} \cdot \frac{2}{|1-z|} + \frac{1}{M} \cdot \frac{2}{|1-z|} + \frac{2}{|1-z|} \cdot \sum_{n=M}^{N-1} \left(\frac{1}{n} - \frac{1}{n+1}\right) \\
&\overset{\text{by CH}}{=} \frac{1}{N} \cdot \frac{2}{|1-z|} + \frac{1}{M} \cdot \frac{2}{|1-z|} + \frac{2}{|1-z|} \cdot \left(\frac{1}{M} - \frac{1}{N}\right) \\
&\leq \frac{1}{N} \cdot \frac{2}{|1-z|} + \frac{1}{M} \cdot \frac{2}{|1-z|} + \frac{1}{M} \cdot \frac{2}{|1-z|} \\
&\overset{\text{by (2)}}{\leq} \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \; .
\end{aligned}$$

Thus the sequence $\left\{\sum_{n=1}^{N} \frac{z^n}{n}\right\}_{N=1}^{\infty}$ is a Cauchy sequence in the complete metric space $\mathbb{C}$ and hence the sequence $\left\{\sum_{n=1}^{N} \frac{z^n}{n}\right\}_{N=1}^{\infty}$ converges. ⟨But to say that a series converges is really just saying that the sequence of partial sums of the series converges.⟩ Thus the series $\sum_{n=1}^{\infty} \frac{z^n}{n}$ converges. □