Review (if needed) lim sup/inf.

Read the handout *Limit Superior (lim sup) and Limit Inferior (lim inf) of sequence* (from $\mathbb{R}$), which is posted on our Math 703/704 homepage. In particular, note the Claims 10 and 12.

Fact. Let $\{s_n\}_{n=1}^{\infty}$ be a sequence in $\mathbb{R}$. Let $\widehat{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$.

Claim 10. There exists a subsequence $\{s_{n_k}\}_{k=1}^{\infty}$ of $\{s_n\}_{n=1}^{\infty}$ s.t. $\lim_{k\to\infty} s_{n_k} = \overline{\lim}_{n\to\infty} s_n \in \widehat{\mathbb{R}}$. Thus, if $\{s_n\}_{n=1}^{\infty}$ is bounded above, then it has a subsequence that converges to an element in $\mathbb{R}$.

Claim 12.

$$\overline{\lim_{n\to\infty}}\, s_n \;=\; \sup\left\{ \lim_{k\to\infty} s_{n_k} \;:\; \{s_{n_k}\}_{k=1}^{\infty} \text{ is a subsequence of } \{s_n\}_{n=1}^{\infty} \text{ s.t. } \lim_{k\to\infty} s_{n_k} \in \widehat{\mathbb{R}} \right\} .$$

Lemma 1. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of nonnegative real numbers. Then

$$\overline{\lim_{n\to\infty}}\, (a_n)^2 \;=\; \left(\overline{\lim_{n\to\infty}}\, a_n\right)^2 . \tag{1}$$

(The equality in (1) is in the extended sense, i.e., one side is infinity if and only if the other side is infinity.)

Proof of Lemma 1. LTGBG. Note that since $\{a_n\}_{n=1}^{\infty}$ and $\left\{(a_n)^2\right\}_{n=1}^{\infty}$ are each bounded below by 0, the $\overline{\lim}_{n\to\infty} a_n$ and $\overline{\lim}_{n\to\infty} (a_n)^2$ exist in $[0,\infty]$.

$\boxed{\geq}$. By Claim 10, there exists a subsequence $\{a_{n_k}\}_k$ whose limit is $\overline{\lim}_{n\to\infty} a_n$, and so

$$\lim_{k\to\infty} a_{n_k} \;=\; \overline{\lim_{n\to\infty}}\, a_n \in [0,\infty] . \tag{2}$$

Thus

$$\lim_{k\to\infty} (a_{n_k})^2 = \lim_{k\to\infty} (a_{n_k}) \cdot (a_{n_k}) \overset{(*)}{=} \left(\lim_{k\to\infty} a_{n_k}\right) \cdot \left(\lim_{k\to\infty} a_{n_k}\right) = \left(\lim_{k\to\infty} a_{n_k}\right)^2 \overset{\text{by (2)}}{=} \left(\overline{\lim_{n\to\infty}}\, a_n\right)^2 ,$$

where $(*)$ holds since $\lim_{k\to\infty} a_{n_k}$ exists in $[0,\infty]$. Thus, $\left(\overline{\lim}_{n\to\infty} a_n\right)^2$ is a subsequential limit point of $\left\{(a_n)^2\right\}_n$. Thus by Claim 12,

$$\left(\overline{\lim_{n\to\infty}}\, a_n\right)^2 \;\leq\; \overline{\lim_{n\to\infty}}\, (a_n)^2 .$$

$\boxed{\leq}$. Since the a_n's are nonnegative, it is equivelent to show

$$\sqrt{\overline{\lim_{n\to\infty}}\, (a_n)^2} \;\leq\; \overline{\lim_{n\to\infty}}\, a_n . \tag{3}$$

By Claim 10, there exists a subsequence $\left\{(a_{n_k})^2\right\}_k$ whose limit is $\overline{\lim}_{n\to\infty} (a_n)^2$, and so

$$\lim_{k\to\infty} (a_{n_k})^2 \;=\; \overline{\lim_{n\to\infty}}\, (a_n)^2 \in [0,\infty] . \tag{4}$$

Thus

$$\lim_{k\to\infty} a_{n_k} = \lim_{k\to\infty} \sqrt{(a_{n_k})^2} \overset{(**)}{=} \sqrt{\lim_{k\to\infty} (a_{n_k})^2} \overset{\text{by (4)}}{=} \sqrt{\overline{\lim_{n\to\infty}}\, (a_n)^2} ,$$

where $(**)$ holds since $\lim_{k\to\infty} (a_{n_k})^2$ exists in $[0,\infty]$ $\langle$and $\sqrt{\cdot}\colon [0,\infty) \to \mathbb{R}$ is continuous with $\lim_{x\to\infty} \sqrt{x} = \infty$.$\rangle$.

Thus, $\sqrt{\overline{\lim}_{n\to\infty} (a_n)^2}$ is a subsequential limit point of $\{a_n\}_n$. Thus by Claim 12, (3) holds. □

Exercise. Let the power series $\sum_{n=0}^{\infty} a_n z^n$ have radius of convergence R

(1) What is the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n (2z)^n$?

(2) What is the radius of convergence of the power series $\sum_{n=0}^{\infty} (a_n)^2 z^n$?

Remark, here $z, a_n \in \mathbb{C}$.

Solution. LTGBG. Since $\sum_{n=0}^{\infty} a_n z^n$ has radius of convergence R,

$$\frac{1}{R} = \overline{\lim_{n\to\infty}} |a_n|^{1/n} \in [0, \infty] \tag{5}$$

where, by convention (and as in class), $\frac{1}{\infty} = 0$ and $\frac{1}{0} = \infty$.

(1). We rewrite the first power series $\sum_{n=0}^{\infty} a_n (2z)^n$ as

$$\sum_{n=0}^{\infty} a_n (2z)^n = \sum_{n=0}^{\infty} (2^n a_n) z^n$$

and compute

$$\overline{\lim_{n\to\infty}} |(2^n a_n)|^{\frac{1}{n}} = 2 \overline{\lim_{n\to\infty}} |a_n|^{\frac{1}{n}} \overset{\text{by (5)}}{=} \frac{2}{R} = \frac{1}{\frac{R}{2}} .$$

Thus $\sum_{n=0}^{\infty} a_n (2z)^n$ has radius of convergence $\frac{R}{2}$.

(2). For the second power series $\sum_{n=0}^{\infty} (a_n)^2 z^n$, we compute

$$\overline{\lim_{n\to\infty}} \left|(a_n)^2\right|^{\frac{1}{n}} = \overline{\lim_{n\to\infty}} \left[|a_n|^{\frac{1}{n}}\right]^2 \overset{\text{by Lemma 1}}{=} \left[\overline{\lim_{n\to\infty}} |a_n|^{\frac{1}{n}}\right]^2 \overset{\text{by (5)}}{=} \left[\frac{1}{R}\right]^2 = \frac{1}{R^2} .$$

Thus $\sum_{n=0}^{\infty} (a_n)^2 z^n$ has radius of convergence R^2. □