Recall

Cauchy-Riemann Equations for $f = u + iv$ are: $u_x = v_y$ and $u_y = -v_x$.
Prop. 4.10. If $f \in H(G)$ and $f'(z) = 0$ for each z in the nonempty open connected subset G of $\mathbb{C}$, then f is constant on G.

Exercise. Let $f \in H(G)$ where G is a nonempty open connected subset of $\mathbb{C}$. Prove the following.

1. If $\operatorname{Re} f$ is constant on G, then f is constant on G.
2. If $\operatorname{Im} f$ is constant on G, then f is constant on G.
3. If $|f|$ is constant on G, then f is constant on G.

Do so without using facts not covered thus far in class. So you may use ideas from the Class Script's Section 1.1-1.3 as well as Prop. 4.10.

Proof's Idea. Let $f \in H(G)$ where G is a nonempty open connected subset of $\mathbb{C}$. As usual, write $f = u + iv$ where $u := \operatorname{Re} f$ and $v := \operatorname{Im} f$. Since $f \in H(G)$, on G: the first order partial derivatives of u and v exist, they satisfy the CR equations

$$u_x = v_y \qquad \text{and} \qquad u_y = -v_x \ , \tag{CReq}$$

and $f' = u_x + iv_x = v_y - iu_y$.

1. Let u be constant on G. Then on G the partials $u_x = 0$ and $u_y = 0$. So $f' = 0$ on G since

$$f' = u_x + iv_x = u_x - iu_y.$$

So ⟨by Prop. 4.10⟩ f is constant on G.

2. Similar to part 1 but using $f' = u_x + iv_x = v_y + iv_x$.
Can also do by applying part 1 to $-if = v - iu \in H(G)$ since $\operatorname{Re}(-if) = \operatorname{Im} f$.

3. Let $c \in \mathbb{C}$. Let $|f(z)| = c$ for each $z \in G$. If $c = 0$, then we are done. So assume $c \neq 0$. Then

$$g(x+iy) := |f(x+iy)|^2 = [u(x,y)]^2 + [v(x,y)]^2 = c^2 \neq 0.$$

Taking the partial derivatives of g w.r.t. x and y we get (on G)

$$2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} = 0 \tag{1}$$

$$2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y} = 0. \tag{2}$$

Using the CR equations we rewrite (2) as

$$-2u\frac{\partial v}{\partial x} + 2v\frac{\partial u}{\partial x} = 0. \tag{3}$$

Muliplying (1) by u gives

$$2u^2\frac{\partial u}{\partial x} + 2uv\frac{\partial v}{\partial x} = 0 \tag{4}$$

Muliplying (3) by v gives

$$-2uv\frac{\partial v}{\partial x} + 2v^2\frac{\partial u}{\partial x} = 0. \tag{5}$$

Adding (4) and (5) gives that on G

$$2\left(u^2+v^2\right)\frac{\partial u}{\partial x} = 0. \tag{6}$$

But $u^2 + v^2 \neq 0$ on G so (6) gives $\frac{\partial u}{\partial x} = 0$ on G. Similarly $\frac{\partial v}{\partial x} = 0$ on G. So $f' = u_x + iv_x = 0$ on G. Since f is holomorohic on the nonempty open connected set G and $f' = 0$ on G, f is constant on G (cf. Prop. 4.10). $\square$