Exercise. Define $u\colon \mathbb{R}^2 \to \mathbb{R}$ by

$$u(x,y) = x^2 - y^2.$$

1. Show that u is harmonic on $\mathbb{C}$.
2. Find $f \in H(\mathbb{C})$ such that $u = \operatorname{Re} f$.

Def. A function $u\colon G \to \mathbb{R}$, where G is an open subset of $\mathbb{R}^2$, is harmonic on G provided:

1. the first and second partials derivatives of u exist and are continuous on G,
2. u satisfies the LaPlace equation $\triangle u = 0$ on G, i.e., $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ on G.

Solution. Define $u\colon \mathbb{R}^2 \to \mathbb{R}$ by

$$u(x,y) = x^2 - y^2.$$

$\boxed{1}$. Since u is a polyomonial on $\mathbb{R}^2$, the first and second partials derivatives of u exist and are continuous on $\mathbb{C}$. Since u_{xx} is the constant function 2 while u_{yy} is the constant function -2. the function u satisfies the Laplace equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ on $\mathbb{C}$. Thus u is harmonic on $\mathbb{C}$.

$\boxed{2}$. We want to find a harmonic conjugate of u on $\mathbb{C}$, call it $v = v(x,y)$. Note if $f = u + iv$ is to be analytic, then f must satisfy the Cauchy-Riemann equations on $\mathbb{C}$, i.e.,

$$u_x = v_y \qquad \text{and} \qquad u_y = -v_x\ . \tag{CReq}$$

Since

$$\begin{aligned} v(x,y) &= \int \left[\frac{\partial v}{\partial y} \right] dy \\ &\overset{\text{(CR)}}{=} \int \left[\frac{\partial u}{\partial x} \right] dy \\ &= \int [\ 2x\ ]\ dy \\ &= 2xy + h(x), \end{aligned} \tag{1}$$

for some funtion $h\colon \mathbb{R} \to \mathbb{R}$, we have that

$$\frac{\partial v}{\partial x} \overset{\text{by (1)}}{=} \frac{\partial}{\partial x} [\ 2xy + h(x)\ ] = 2y + h'(x)\ . \tag{2}$$

On the other hand

$$\frac{\partial v}{\partial x} \overset{\text{CR}}{=} \frac{-\partial u}{\partial y} = \frac{-\partial}{\partial y}\left(x^2 - y^2\right) = 2y. \tag{3}$$

By (2) and (3), the function h' is identically zero. So there is $c \in \mathbb{R}$ such that $h(x) = c$ and

$$v(x,y) = 2xy + c,$$

▷ Possibities for $f \in H(\mathbb{C})$ s.t. $u = \operatorname{Re} f$. Pick your favorite $c \in \mathbb{R}$ (I'll pick $c = 17$). Then take

$$f(x+iy) = \left[x^2 - y^2\right] + i\,[\ 2xy + 17\ ],$$

which can be expressed as the clearly analytic function

$$f(z) = z^2 + 17i.$$