Exercise. Let G be an open subset of $\mathbb{C}$ and $f \in H(G)$. Define

$$G^* := \{z \in \mathbb{C} \colon \overline{z} \in G\}$$
$$f^*(z) := \overline{f(\overline{z})} \qquad \text{for } z \in G^* .$$

Note (i.e., you need not show) that G^* is open in $\mathbb{C}$.

1. Show that $f^* \in H(G^*)$.
2. Express $(f^*)'$ in terms of f'.

Hint. Does your solution to part 2 make sense to you geometrically?

Solution Way 1. Fix $z \in G^*$. Note that for each $h \in \mathbb{C}$ of sufficiently small modulus, to be precise so that $z + h \in G^*$,

$$\frac{f^*(z+h) - f^*(z)}{h} = \frac{\overline{f\left(\overline{z+h}\right)} - \overline{f(\overline{z})}}{h} = \frac{\overline{f\left(\overline{z}+\overline{h}\right) - f(\overline{z})}}{h} = \overline{\left(\frac{f\left(\overline{z}+\overline{h}\right) - f(\overline{z})}{\overline{h}}\right)} .$$

Thus

$$\lim_{h\to 0} \frac{f^*(z+h) - f^*(z)}{h} = \overline{\lim_{h\to 0} \left(\frac{f\left(\overline{z}+\overline{h}\right) - f(\overline{z})}{\overline{h}}\right)} = \overline{f'(\overline{z})} .$$

Thus $f^* \in H(G^*)$ and

$$\boxed{(f^*)'(z) = \overline{f'(\overline{z})}} . \tag{1}$$

□

Solution Way 2. Let

$$u := \operatorname{Re} f \quad \text{and} \quad v := \operatorname{Im} f \quad \text{so} \quad f = u + iv$$
$$u^* := \operatorname{Re} f^* \quad \text{and} \quad v^* := \operatorname{Im} f^* \quad \text{so} \quad f^* = u^* + iv^* .$$

Fix $z = x + iy \in G^*$. Simple (you should do) calculations show that

$$u^*(x,y) = u(x,-y) \qquad \text{and} \qquad v^*(x,y) = -v(x,-y) .$$

Thus,

(1) applying the chain rule properly

(2) using the fact that u and v satisfy the CR equations at $(x,-y)$

gives

$$\begin{aligned} \frac{\partial u^*}{\partial x}(x,y) &\overset{(1)}{=} \frac{\partial u}{\partial x}(x,-y) \overset{(2)}{=} \frac{\partial v}{\partial y}(x,-y) \overset{(1)}{=} (-1)(-1)\frac{\partial v^*}{\partial y}(x,y) \\ \frac{\partial u^*}{\partial y}(x,y) &\overset{(1)}{=} -\frac{\partial u}{\partial y}(x,-y) \overset{(2)}{=} \frac{\partial v}{\partial x}(x,-y) \overset{(1)}{=} \frac{\partial v^*}{\partial x}(x,y) . \end{aligned} \tag{2}$$

Thus u^* and v^* satisfy the CR equations. So $f^* \in H(G^*)$. Since if a function g is differentiable at $z_0 = x_0 + iy_0$, then $g'(z_0) = (\operatorname{Re} g)_x (x_0, y_0) + i (\operatorname{Im} g)_x (x_0, y_0)$. So from the above two calculations in (2), the formula in (1) holds. □