Exercise. Define $f\colon \mathbb{C} \to \mathbb{C}$ and $u, v\colon \mathbb{R}^2 \to \mathbb{R}$ by

$$\begin{aligned} f(z) &:= \sqrt{|xy|} \qquad \text{where } x := \operatorname{Re} z \quad \text{and} \quad y := \operatorname{Im} z \\ u(x,y) &= \operatorname{Re} f(x+iy) \\ v(x,y) &= \operatorname{Im} f(x+iy) \ . \end{aligned}$$

Show that

1. u and v satisfies the Cauchy Riemann equations at $(x,y) = (0,0)$
2. f is not differentiable at $z = 0$.

Recall that (complex) differentiable means differentiable as defined in Class Script, p. 4, Def. I.3.1.
Remark. As usual in such a setting, we write $z = x+iy$ with $x, y \in \mathbb{R}$ and let $u(x,y) = \operatorname{Re}(f(x,y))$ and $v(x,y) = \operatorname{Im}(f(x,y))$. Thus $f(x+iy) = u(x,y) + iv(x,y)$.

Recall. The Cauchy-Riemann Equations (CReq) for f are

$$u_x = v_y \qquad \text{and} \qquad u_y = -v_x \ . \tag{CReq}$$

Recall some Big Theorems

Let $f\colon G \to \mathbb{C}$ where G is an open subset of $\mathbb{C}$. Fix $z_0 \in G$.

- ▷ Diff⇒CR. Let f is differentiable at z_0. Then the CR equations for f are satisfied at z_0.
- ▷ CR+more⇒Diff. Let the CR equations for f be satisfied at z_0 and, furthermore, the first partial derivatives u_x, v_y, u_y, and v_x:
  1. exist in some neighborhood $N_\varepsilon(z_0)$ of z_0
  2. be continuous at z_0.

  Then f is differentiable at z_0.
- ▷ If f is differentiable at $z_0 = x_0 + iy_0$, then $f'(z_0) = u_x(x_0,y_0) + iv_x(x_0,y_0) = v_y(x_0,y_0) - iu_y(x_0,y_0)$.

Solution. As usual in such a setting, we write $z = x+iy$ with $x, y \in \mathbb{R}$ and let $u(x,y) = \operatorname{Re}(f(x,y))$ and $v(x,y) = \operatorname{Im}(f(x,y))$. Thus $f(x+iy) = u(x,y) + iv(x,y)$. Note that in this problem, for each $(x,y) \in \mathbb{R}^2$.

$$u(x,y) = \sqrt{|xy|} \qquad \text{and} \qquad v(x,y) = 0 \qquad \text{and} \qquad \frac{\partial v}{\partial x}(x,y) = 0 = \frac{\partial v}{\partial y}(x,y) \ .$$

To find $\frac{\partial u}{\partial x}(0,0)$, we use the definiton:

$$\frac{\partial u}{\partial x}(0,0) = \lim_{\substack{h\to 0 \\ h\in\mathbb{R}}} \frac{u(h,0) - u(0,0)}{h} = \lim_{\substack{h\to 0 \\ h\in\mathbb{R}}} \frac{0-0}{h} = \lim_{\substack{h\to 0 \\ h\in\mathbb{R}}} 0 = 0 \ .$$

Similarly, $\frac{\partial u}{\partial y}(0,0) = 0$. Thus

$$\frac{\partial u}{\partial x}(0,0) = 0 = \frac{\partial v}{\partial y}(0,0) \qquad \text{and} \qquad \frac{\partial u}{\partial y}(0,0) = 0 = \frac{-\partial v}{\partial x}(0,0)$$

and so the Cauchy-Riemann equations hold at $z = 0$. However

$$\frac{f(0 + (h + ih)) - f(0)}{h + ih} = \frac{\sqrt{|h \cdot h|}}{h(1 + i)} = \frac{|h|}{h} \cdot \frac{1}{1 + i} = \frac{|h|}{h}\left(\frac{1}{2} + i\frac{1}{2}\right)$$

and so

$$\lim_{\substack{h \to 0^+ \\ h \in \mathbb{R}}} \frac{f(0 + (h + ih)) - f(0)}{h + ih} = \left(\frac{1}{2} + i\frac{1}{2}\right)$$

while

$$\lim_{\substack{h \to 0^- \\ h \in \mathbb{R}}} \frac{f(0 + (h + ih)) - f(0)}{h + ih} = -\left(\frac{1}{2} + i\frac{1}{2}\right) .$$

Thus

$$\lim_{\substack{h \to 0 \\ h \in \mathbb{C}}} \frac{f(0 + h) - f(0)}{h}$$

does not exist. So f is not differentiable at $z = 0$. □