Exercise. Prove the following functions are nowhere (complex) differentiable on $\mathbb{C}$.
Recall that (complex) differentiable means differentiable as defined in Class Script, p. 4, Def. I.3.1.
(a) $f(z) = \operatorname{Re} z$.
(b) $f(z) = |z|$.

Recall that (complex) differentiable means differentiable as defined in Class Script, p. 4, Def. I.3.1.
Def. A function $f\colon G \to \mathbb{C}$, where G is an open subset of $\mathbb{C}$, is nowhere differentiable on G provided f is not differentiable at z for each $z \in G$.
Remark. As usual in such a setting, we write $z = x+iy$ with $x, y \in \mathbb{R}$ and let $u(x,y) = \operatorname{Re}(f(x,y))$ and $v(x,y) = \operatorname{Im}(f(x,y))$. Thus $f(x+iy) = u(x,y) + iv(x,y)$.

Recall. The Cauchy-Riemann Equations (CReq) for f are

$$u_x = v_y \quad \text{and} \quad u_y = -v_x \,. \tag{CReq}$$

Recall some Big Theorems

Let $f\colon G \to \mathbb{C}$ where G is an open subset of $\mathbb{C}$. Fix $z_0 \in G$.

▷ Diff⇒CR. Let f is differentiable at z_0. Then the CR equations for f are satisfied at z_0.

▷ CR+more⇒Diff. Let the CR equations for f be satisfied at z_0 and, furthermore, the first partial derivatives u_x, v_y, u_y, and v_x:
1. exist in some neighborhood $N_\varepsilon(z_0)$ of z_0
2. be continuous at z_0.

Then f is differentiable at z_0.

▷ If f is differentiable at $z_0 = x_0 + iy_0$, then $f'(z_0) = u_x(x_0,y_0) + iv_x(x_0,y_0) = v_y(x_0,y_0) - iu_y(x_0,y_0)$.

Proof of (a). Note $u(x,y) = x$ and $v(x,y) = 0$. Clearly $u_x(x,y) = 1$ and $v_y(x,y) = 0$ for each $z = x+iy \in \mathbb{C}$. So the CR equations hold nowhere. So f is nowhere differentiable. □

Proof of (b). Note $u(x,y) = \sqrt{x^2+y^2}$ and $v(x,y) = 0$.
Case $x+iy \neq 0$. Thus $x^2+y^2 \neq 0$ and so

$$u_x(x,y) = \frac{x}{\sqrt{x^2+y^2}}, \quad u_y(x,y) = \frac{y}{\sqrt{x^2+y^2}}, \quad v_x(x,y) = 0, \text{ and } v_y(x,y) = 0.$$

For f to satisfy the CR equations at $x+iy$, we must have $u_x(x,y) = 0$ and $u_y(x,y) = 0$, which would give that $x = 0 = y$. This cannot be since $x+iy \neq 0$. So if $z \in \mathbb{C} \setminus \{0\}$, then f is not differentiable at z.
Case $x+iy = 0$. To see that u_x does not even exist at $(x,y) = (0,0)$, note that, for $h \in \mathbb{R} \setminus \{0\}$,

$$\frac{u(0+h,0) - u(0,0)}{h} = \frac{|h|}{h} \longrightarrow \begin{cases} 1 & \text{as } h \to 0^+ \\ -1 & \text{as } h \to 0^- \,. \end{cases}$$

Since u_x does not even exist at $(x,y) = (0,0)$, the function f is not differentiable at $z = 0$.
We conclude that (complex) differentiability of f fails everywhere. □