**Observation 1**. Let $z \in \mathbb{C}$. TFAE, as easily seen by writing $z = x + iy \in \mathbb{C}$ where $x = \operatorname{Re} z \in \mathbb{R}$ and $y = \operatorname{Im} z \in \mathbb{R}$.

(1.1) $z \geq 0$ (a convenient way to indicate that $z$ is real and nonnegative, i.e. $z = \operatorname{Re} z \geq 0$)

(1.2) $z = |z|$

(1.3) $\operatorname{Re} z = |z|$

**Observation 2**. Let $z_1, z_2 \in \mathbb{C}$. TFAE. (think what this is saying geometrically)

(2.1) $z_1 \overline{z_2} \geq 0$

(2.2) $z_1 \, \overline{z_2} = |z_1| \, |z_2|$

(2.3) $\operatorname{Re} z_1 \overline{z_2} = |z_1 \overline{z_2}|$

(2.4) $[\ z_2 = 0\ ]$ or $\left[\ \frac{z_1}{z_2} \geq 0\ \right]$

(2.5) $[\ z_2 = 0\ ]$ or $[\ z_1 = \lambda z_2$ for some $\lambda \in [0, \infty)\ ]$

**Theorem 1**. [Triangle Inequality (with equality)] Let $n \in \mathbb{N}$ and $z_1, \ldots, z_n$ from $\mathbb{C}$. Then

$$|z_1 + \ldots + z_n| \ \leq \ |z_1| + \ldots + |z_n| \tag{1}$$

and equality holds in (1) if and only if

$$z_j \, \overline{z_k} \ = \ |z_j| \, |z_k| \qquad \text{for each} \qquad j, k \in \mathbb{N}^{\leq n} \text{ with } j \neq k. \tag{2}$$

An equivalent formuation of (2) is

$$z_j \, \overline{z_k} \ = \ |z_j| \, |z_k| \qquad \text{for each} \quad j, k \in \mathbb{N}^{\leq n}. \tag{2$'$}$$

*Proof.* Let $n \in \mathbb{N}$. The equivalence of (2) and (2′) follows from $z\overline{z} = |z|^2$ for any $z \in \mathbb{C}$. Theorem 1 clearly holds when $n = 1$ for any $z_1 \in \mathbb{C}$. Thus we assume $n \geq 1$.

First we show inequality (1) by induction. Let $n = 2$. Fix any $z_1, z_2 \in \mathbb{C}$. Then (1) holds aince

$$\begin{aligned}
|z_1 + z_2|^2 &= (z_1 + z_2)\, \overline{(z_1 + z_2)} \\
&= (z_1 + z_2)\, (\overline{z_1} + \overline{z_2}) \\
&= z_1\, \overline{z_1} + z_2\, \overline{z_2} + z_1\, \overline{z_2} + \overline{z_1}\, z_2 \\
&= |z_1|^2 + |z_2|^2 + z_1\, \overline{z_2} + \overline{z_1\, \overline{z_2}} \\
&= |z_1|^2 + |z_2|^2 + 2\operatorname{Re}\, (z_1\, \overline{z_2}) \\
&\overset{(*)}{\leq} |z_1|^2 + |z_2|^2 + 2\, |z_1\, \overline{z_2}| \\
&= |z_1|^2 + |z_2|^2 + 2\, |z_1|\ |z_2| \\
&= (|z_1| + |z_2|)^2 .
\end{aligned} \tag{3}$$

Now fix $n \in \mathbb{N}^{\geq 2}$. Assume that Theorem 1 holds for any collection of $m \in \mathbb{N}$ complex numbers where $m \leq n$. Fix $z_1, \ldots, z_{n+1}$ from $\mathbb{C}$. Then

$$\left| \sum_{j=1}^{n+1} z_j \right| \ = \ \left| \left( \sum_{j=1}^{n} z_j \right) + z_{n+1} \right| \ \leq \ \left| \sum_{j=1}^{n} z_j \right| + |z_{n+1}| \ \leq \ \left( \sum_{j=1}^{n} |z_j| \right) + |z_{n+1}| \ = \ \sum_{j=1}^{n=1} |z_n| \, .$$

Thus (1) holds.

Thus, for any $n \in \mathbb{N}$ and $z_1, \ldots, z_n$ from $\mathbb{C}$, inequality (1) holds.

The remanindער of the proof is the next Exercise. □

You are **strongly** encouraged to work in groups, following the procedure as in homework MS09.

**Exercise pCA 5.** Variant of 3.1.24.3 (p. 173).

Read section 3.1. Then finish the proof of Theorem 1 from the previous page.