Defining the Exponential Function. Using the real exponential function $e^{(\cdot)} \colon \mathbb{R} \to \mathbb{R}$, the complex exponential function $\exp(\cdot) : \mathbb{C} \to \mathbb{C}$ is defined by

$$\exp(x+iy) := e^x \cos y + ie^x \sin y \overset{\text{i.e.}}{=} e^x (\cos y + i \sin y) \quad , \; x, y \in \mathbb{R}. \tag{1}$$

The complex exponential function restricted to $\mathbb{R}$ agrees the the usual real exponential fnc. since

$$\exp(x) = \exp(x+i0) = e^x \cos 0 + ie^x \sin 0 = e^x \quad , \; x \in \mathbb{R}. \tag{2}$$

Thus we often write $\exp(x+iy)$ by e^{x+iy}.

Defining Trigomonetric Functions. From (1) it follows that if $x \in \mathbb{R}$ then

$$e^{ix} + e^{-ix} = 2\cos x \qquad \text{and} \qquad e^{ix} - e^{-ix} = 2i \sin x \quad , \; z \in \mathbb{R}. \tag{3}$$

Motivated by (3), the (complex) sine and cosine functions (from $\mathbb{C}$ to $\mathbb{C}$) are defined by

$$\cos z := \frac{e^{iz} + e^{-iz}}{2} \qquad \text{and} \qquad \sin z := \frac{e^{iz} - e^{-iz}}{2i} \quad , \; z \in \mathbb{C}. \tag{4}$$

Defining Hyperbolic Functions. Motivated by their definition for a real number, the (complex) hyperbolic sine and cosine functions (from $\mathbb{C}$ to $\mathbb{C}$) are defined by

$$\cosh z := \frac{e^{z} + e^{-z}}{2} \qquad \text{and} \qquad \sinh z := \frac{e^{z} - e^{-z}}{2} \quad , \; z \in \mathbb{C}. \tag{5}$$

Good Reference. [BC] Brown and Churchill, *Complex Variables and Applications.* (any edition). From Chapter 3, read sections:

§23. The Exponential Function (p. 65-68)
§24. Trigonometric Functions (p. 69-72)
§25. Hyperolic Functions (p. 72-75).

Concentrate on the definitions and properities of these functions, ignoring for now the parts about derivatives/entire. The above reference shows that these functions are defined on the whole of $\mathbb{C}$ in such a way that

(1) when a function's domain is restricted from $\mathbb{C}$ to $\mathbb{R}$, the resulting function agrees with the function (of the same name) from $\mathbb{R}$ to $\mathbb{R}$ that we know from calculus
(2) many identities/properties which we know from the $\mathbb{R}$-version extend to the $\mathbb{C}$-version (e.g., $e^{z_1} e^{z_2} = e^{z_1 + z_2}$ for each $z_z, z_2 \in \mathbb{C}$).

You should have a working knowledge of the properties/identities of the complex exponential, sine, and cosine functions. One major difference between the complex and real exponential functions is that the complex exponential function is periodic with a pure imaginary period of $2\pi i$, i.e.,

$$\exp(z + 2\pi i) = \exp(z), \quad , \; z \in \mathbb{C}. \tag{6}$$

Thus we will have to take care when defining a complex version of an "'inverse" of the complex exponential (i.e., the complex log).

You are **strongly** encouraged to work in groups, following the procedure as in homework MS09.

Exercise pCA 4. Find all the solutions of the equation $\sin z = 3$, expressing your solution(s) in the form $a + ib$ with $a, b \in \mathbb{R}$.