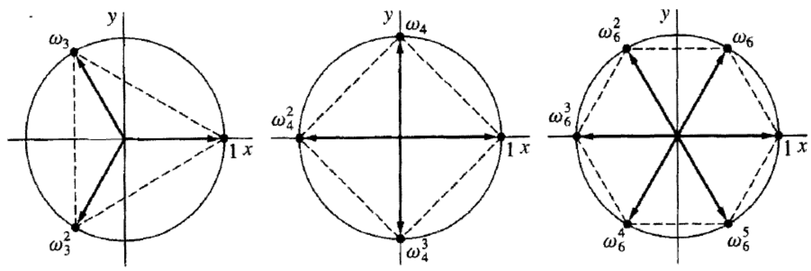
Definition. Following convention, for $n \in \mathbb{N}$, we set

$$\omega_n := \exp\left(i\,\frac{2\pi}{n}\right) \overset{\text{i.e.}}{=} e^{i\,\frac{2\pi}{n}}.$$

The n (distinct) n^{th} roots of unity (where ω_n^k denotes $(\omega_n)^k$) are

$$\left\{ \omega_n^1\,,\ \omega_n^2\,,\ \ldots\,,\ \omega_n^{n-1}\,,\ \omega_n^n\left(\overset{\text{i.e}}{=}1\right) \right\}.$$

Informative. Compute, and draw on the unit circle, the n (distinct) n^{th} roots of unity for $n = 1, 2, 3, \ldots$ (for enough n's until you see the pattern … below $n = 3, 4, 6$ are illustrated).



Note. Any n^{th} root of unity is a solution to the equation $z^n = 1$. Are there more? ⟨NO, as next Thm. shows.⟩
Key Result. The next theorem gives that, for ⟨fixed⟩ $r_0 > 0$ and $\theta_0 \in \mathbb{R}$, the solution set to the equation

$$z^n = r_0 e^{i\theta_0}$$

is the set (of n distinct elements)

$$\left\{ \sqrt[n]{r_0}\ \left(e^{i\theta_0}\right)^{1/n}\ \left(e^{i2\pi k}\right)^{1/n} : k = 0, 1, 2, \ldots, n-1 \right\}.$$

Theorem. *Let $n \in \mathbb{N}$ and $z_0 = r_0 e^{i\theta_0} \in \mathbb{C} \setminus \{0\}$ with $r_0 > 0$ and $\theta \in \mathbb{R}$.* ⟨so θ_0 is any element from *arg* z_0⟩. *Then the solution set of the equation*

$$z^n = z_0$$

is the set (of n distinct elements)

$$\left\{ \sqrt[n]{r_0}\ \exp\left[\frac{i}{n}\Big(\theta_0 + 2\pi k\Big)\right] \in \mathbb{C}\colon k = 0, 1, 2, \ldots, n-1 \right\} \overset{\text{i.e.}}{=} \left\{ \sqrt[n]{|z_0|}\, e^{i\,\frac{\theta_0}{n}}\ (\omega_n)^k \in \mathbb{C}\colon k = 0, 1, 2, \ldots, n-1 \right\}$$

where $w_n = e^{i\,\frac{2\pi}{n}}$. Furthermore, if $c \in \mathbb{C}$ is any solution to $z^n = z_0$, then

$$\left\{ c\,,\ c\omega_n^1\,,\ c\omega_n^2\,,\ c\omega_n^3\,,\ \ldots\,,\ c\omega_n^{n-1} \right\}$$

is a solution set to $z^n = z_0$. ⟨$c = c\omega_n^0$⟩. *If $-\pi < \theta_0 \le \pi$* ⟨*i.e., θ_0 is the principal value of the argument of $r_0 e^{i\theta_0}$*⟩, *then $\sqrt[n]{r_0}\, e^{i\,\frac{\theta_0}{n}}$ is called the principal n^{th} root of z_0.*

Proof's key calculation. LTGBG. Then $\sqrt[n]{r_0} \in \mathbb{R}^{>0}$ since $r_0 \in \mathbb{R}^{>0}$. Then TFAE.

$$z_0 = z^n$$

$$r_0 e^{i\theta_0} = \left(re^{i\theta}\right)^n$$

$$r_0 e^{i(\theta_0 + 2\pi\,k)} = r^n e^{in\theta} \qquad \text{for any } k \in \mathbb{Z}$$

$$r = \sqrt[n]{r_0} \qquad \text{and} \qquad \theta = \frac{\theta_0}{n} + \frac{2\pi k}{n} \qquad \text{for any } k \in \mathbb{Z}$$

$$z = \sqrt[n]{r_0}\ \exp\left[i\left(\frac{\theta_0}{n} + \frac{2\pi k}{n}\right)\right] \qquad \text{for any } k \in \mathbb{Z}$$

$$z \in \left\{ \sqrt[n]{r_0} \exp\left[i\left(\frac{\theta_0}{n} + \frac{2\pi k}{n}\right)\right] \in \mathbb{C}\colon k = 0, 1, 2, \ldots, n-1 \right\}\ . \qquad (☺)$$

Lesson. Need to take care in taking n^{th} roots of complex numbers. The n^{th} roots of a (nonzero) complex number is a set with n distinct elements.
Reference. *Complex Variables and Appl.* by Brown and Churchill (Ch.1's §: *Roots of Complex Numbers*).

You are **strongly** encouraged to work in groups, following the procedure as in homework MS09.

Exercise pCA 2. Solve $z^2 - 4z + (4 + 2i) = 0$. Express your solution(s) in the form $a + ib$ with $a, b \in \mathbb{R}$.