You learn a lot talking math with others. Thus you are **strongly** encouraged to work in groups (up to size 17) on homework. A group is to come to an agreement of the finished paper. Over Blackboard, ONE group member (e.g., Bella) should submit the finished paper while each of the other group members (as so that I can return a commented graded paper to you) should just pull up the assignment on Blackboard and write a note in the white `Comment` box that, e.g., Bella submitted my paper.

Metric Space Exercise 9. Variant of 2.2.29.1 (p. 109).

Let $1 \le p \le \infty$ and $n \in \mathbb{N}$. Let (X_i, d_i) be a metric space for each $i \in \mathbb{N}^{\le N} := \mathbb{N} \cap [1, N]$. Set

$$\bigoplus_{i=1}^{N} X_i := \left\{ \{x_i\}_{i=1}^{N} : x_i \in X_i \text{ for each } i \in \mathbb{N}^{\le N} \right\}. \tag{1}$$

Define $d_p \colon \bigoplus_{i=1}^{N} X_i \ \times \ \bigoplus_{i=1}^{N} X_i \to \mathbb{R}$ by

$$d_p\left(\{x_i\}_{i=1}^{N}, \{y_i\}_{i=1}^{N} \right) := \begin{cases} \left[\sum_{i=1}^{N} [\, d_i(x_i, y_i)\,]^p \right]^{1/p} & \text{, if } 1 \le p < \infty \\ \sup_{1 \le i \le N} d_i(x_i, y_i) & \text{, if } p = \infty. \end{cases}$$

Using our previously defined $\left(\mathbb{R}^N, \|\cdot\|_{\ell_p} \right)$, where

$$\left\| \{b_i\}_{i=1}^{N}, \right\|_{\ell_p} := \begin{cases} \left[\sum_{i=1}^{N} |b_i|^p \right]^{1/p} & \text{, if } 1 \le p < \infty \\ \sup_{i \in \mathbb{N}} |b_i| & \text{, if } p = \infty, \end{cases}$$

we see that

$$d_p\left(\{x_i\}_{i=1}^{N}, \{y_i\}_{i=1}^{N} \right) = \| \{d_i(x_i, y_i)\}_{i=1}^{n} \|_{\ell_p}.$$

MS 9a. Show that d_p is a metric on $\bigoplus_{i=1}^{N} X_i$. Hint: Minkowski's inequality.

MS 9b. Show that if each (X_i, d_i) is complete, then $\left(\bigoplus_{i=1}^{N} X_i, d_p \right)$ is complete.

Hint: A sequence $\{x^n\}_{n=1}^{\infty}$ from $\bigoplus_{i=1}^{N} X_i$ can be envisioned as, where $x^n = \{x_i^n\}_{i=1}^{N} = \{x_1^n, x_2^n, x_3^n, \dots x_N^n\}$,

$$\begin{aligned} x^1 &= \left\{ x_i^1 \right\}_{i=1}^{N} = \{x_1^1, x_2^1, x_3^1, \dots x_N^1\} \\ x^2 &= \left\{ x_i^2 \right\}_{i=1}^{N} = \{x_1^2, x_2^1, x_3^2, \dots x_N^2\} \\ x^3 &= \left\{ x_i^3 \right\}_{i=1}^{N} = \{x_1^3, x_2^3, x_3^3, \dots x_N^3\} \\ &\vdots \\ x^n &= \{x_i^n\}_{i=1}^{N} = \{x_1^n, x_2^n, x_3^n, \dots x_N^n\} \\ &\downarrow \langle \text{as } n \to \infty, \text{ want} \rangle \\ x^0 &= \left\{ x_i^0 \right\}_{i=1}^{N} = \{x_1^0, x_2^0, x_3^0, \dots x_N^0\}. \end{aligned}$$