Pins: 100
Your Last Names

Math 703

Due Date: Fri. 9/25 at 11:59pm. HW: MS04

You learn a lot talking math with others. Thus you are **strongly** encouraged to work in groups (up to size 17) on homework. A group is to come to an agreement of the finished paper. Over Blackboard, ONE group member (e.g., Bella) should submit the finished paper while each of the other group members (as so that I can return a commented graded paper to you) should just pull up the assignment on Blackboard and write a note in the white `Comment` box that, e.g., Bella submitted my paper.

Throughout this Exercise, I is a nonempty closed bounded interval of $\mathbb{R}$.

Since I is a closed bounded interval, $C(I, \mathbb{R}) = \mathscr{C}(I, \mathbb{R}) \subset \mathscr{R}(I, \mathbb{R})$ where:

$$C(I, \mathbb{R}) := \{f : I \to \mathbb{R} \mid f \text{ is continuous on } I\}.$$

$$\mathscr{C}(I, \mathbb{R}) := \{f : I \to \mathbb{R} \mid f \text{ is bounded and continuous on } I\}$$

$$\mathscr{R}(I, \mathbb{R}) := \{f : I \to \mathbb{R} \mid f \text{ is Riemann integrable over } I\}.$$

Note that $C(I, \mathbb{R})$ (resp. $\mathscr{C}(I, \mathbb{R})$, $\mathscr{R}(I, \mathbb{R})$) is a *vector space* over the field $\mathbb{R}$.

Now let $1 \le p < \infty$. If $h \in C(I, \mathbb{R})$ then $|h|^p \in C(I, \mathbb{R}) \subset \mathscr{R}(I, \mathbb{R})$. Thus we can define the function $\|\cdot\|_p : C(I, \mathbb{R}) \to \mathbb{R}$ by

$$\|f\|_p := \left[\int_I |f(t)|^p \right]^{1/p} \quad , \text{ for } f \in C(I, \mathbb{R})$$

and corresponding function $d_p : C(I, \mathbb{R}) \times C(I, \mathbb{R}) \to \mathbb{R}$ by

$$d_p(f, g) := \left[\int_I |f(t) - g(t)|^p \right]^{1/p} \quad , \text{ for } f, g \in C(I, \mathbb{R})$$

Note $d_p(f, g) = \|f - g\|_p$.

Recall that $\|\cdot\| : X \to \mathbb{R}$ is a <u>norm</u> on a vector space X (over the field $\mathbb{K}$ of $\mathbb{R}$ or $\mathbb{C}$) provided

(N1) $\|x\| = 0$ if and only if $x = 0$,

(N2) $\|\alpha x\| = |\alpha| \|x\|$,

(N3) $\|x + y\| \le \|x\| + \|y\|$

for each $x, y \in X$ and $\alpha \in \mathbb{K}$.

Metric Space Exercise 4. Variant of 2.1.45.2 (p. 90).

This is the *function* version of the *sequence* Exercise 2.1.45.1 (hwMS2). You may use, without (re)proving, anything from hwMS2.

Metric Space Exercise 4a. Let $\|\cdot\| : X \to \mathbb{R}$ is a norm on a vector space X (over $\mathbb{R}$ or $\mathbb{C}$).

Let $f \in C(I, \mathbb{R})$ and $\alpha \in \mathbb{R}$. Let $1 \le p < \infty$.

Show that $\|x\| \ge 0$ for all $x \in X$ by using properties (N1), (N2), and (N3).

Show that $\|f\|_p = 0$ if and only if $f = 01_I$ ⟨i.e., if and only if $f(t) = 0$ for all $t \in I$⟩.

Show that $\|\alpha f\|_p = |\alpha| \|f\|_p$.

Metric Space Exercise 4b. (Hölder's Inequality for Continuous Functions). Let $1 < p < \infty$.

Let $f, g \in C(I, \mathbb{R})$ and $p' \in (1, \infty)$ be the *conjugate exponent* of p (thus $\frac{1}{p} + \frac{1}{p'} = 1$). Show

$$\|fg\|_1 \leq \|f\|_p \|g\|_{p'} \tag{H}$$

that is

$$\int_I |f(t) g(t)| \, dt \leq \left[\int_I |f(t)|^p \, dt \right]^{1/p} \left[\int_I |g(t)|^{p'} \, dt \right]^{1/p'}.$$

Metric Space Exercise 4c. (Minkowski's Inequality for Cnt. Functions). Let $1 < p < \infty$.

Let $f, g \in C(I, \mathbb{R})$. Show

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p \tag{M}$$

that is

$$\left[\int_I |f(t) + g(t)|^p \, dt \right]^{1/p} \leq \left[\int_I |f(t)|^p \, dt \right]^{1/p} + \left[\int_I |g(t)|^p \, dt \right]^{1/p}.$$

Does Minkowski's Inequality hold when $p = 1$?

Hint for $1 < p < \infty$. If $a, b \in \mathbb{R}$ and $1 < p < \infty$, then $0 \neq p - 1$ ⟨so $|0 + 0|^{p-1}$ makes sense⟩ and so

$$|a + b|^p = |a + b|^1 \, |a + b|^{p-1}$$

$$\leq [\, |a| + |b| \,] \, |a + b|^{p-1}$$

$$= |a| \, |a + b|^{p-1} + |b| \, |a + b|^{p-1}.$$

Metric Space Exercise 4d. Let $1 \leq p \leq \infty$.

Conclude that $\left(C(I, \mathbb{R}), \|\cdot\|_p \right)$ is a normed vector space.

Conclude that $(C(I, \mathbb{R}), d_p)$ is a metric space.