You learn a lot talking math with others. Thus you are **strongly** encouraged to work in groups (up to size 17) on homework. A group is to come to an agreement of the finished paper and then each group member should submit over Blackboard the identical finished paper. Follow the instructions at the top of the LaTex file to but all PINs and Names on the paper. A graded copy of the group's finished paper will be returned to each group member.

Let $N \in \mathbb{N}$. Define $d_p \colon \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ by

$$d_p\left(\{x_i\}_{i=1}^N, \{y_i\}_{i=1}^N\right) := \begin{cases} \left[\sum_{i=1}^N |x_i - y_1|^p\right]^{\frac{1}{p}} & \text{, if } 1 \le p < \infty \\ \sup_{1\le i\le N} |x_i - y_i| & \text{, if } p = \infty \end{cases} \tag{1}$$

and $\|\cdot\|_{\ell_p} \colon \mathbb{R}^N \to \mathbb{R}$ by

$$\left\|\{x_i\}_{i=1}^N\right\|_{\ell_p} := \begin{cases} \left[\sum_{i=1}^N |x_i|^p\right]^{\frac{1}{p}} & \text{, if } 1 \le p < \infty \\ \sup_{1\le i\le N} |x_i| & \text{, if } p = \infty \end{cases} \tag{2}$$

for $x := \{x_i\}_{i=1}^N$ and $y := \{y_i\}_{i=1}^N$ in $\mathbb{R}^N$. Note $d_p(x,y) = \|x-y\|_{\ell_p}$ where $\{x_i\}_{i=1}^N - \{y_i\}_{i=1}^N := \{x_i - y_i\}_{i=1}^N$. In class we observed that $\left(R^N, d_p\right)$ is a metric space when p is 1 or ∞. The goal is this exercies is to extend this fact to $1 \le p \le \infty$. We shall use the concept of convex functions, which is useful in showing many inequalities in mathematics. This Exercise set is a variant of Exercise 2.1.45.1 (page 89). The actual statements of the exercise's parts are toward the end of the file.

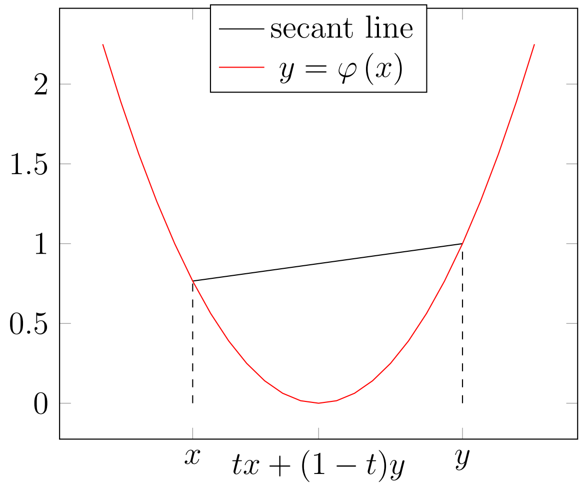
Notation 1. Throughout this Exercise, $N \in \mathbb{N}$ is fixed and

- $(a,b) = I \subset \mathbb{R}$ where $a, b \in \mathbb{R} \cup \{\pm\infty\}$
- $\varphi \colon I \to \mathbb{R}$ is a function
- $p \in (1,\infty)$ and its *conjugate exponent* $p' \in (1,\infty)$ is defined by $\frac{1}{p} + \frac{1}{p'} = 1$.

Definition 2. The function $\varphi \colon I \to \mathbb{R}$ is *convex* provided

$$[x, y \in I \text{ and } t \in (0,1)] \implies \varphi(tx + (1-t)y) \le t\varphi(x) + (1-t)\varphi(y) . \tag{3}$$

Graphically, for a convex function φ on I and any $(x,y) \subset I$, on the interval (x,y), the graph of the function φ on the interval (x,y) lie below the secant line through $(x, \varphi(x))$ and $(y, \varphi(y))$.



Proposition 3. *Let the second derivative of $f\colon I \to \mathbb{R}$ exist at each point of I. Then f is a convex function on I if and only if $f''(x) \geq 0$ for each $x \in I$.*

A proof of Proposition 3 is posted on the handout page.

Lemma 4. *Let $\varphi\colon I \to \mathbb{R}$ be convex. Then:*

$$x_i \in I \quad , \quad t_i \in (0,1) \quad , \quad \sum_{i=1}^{n} t_i = 1 \quad \Longrightarrow \quad \varphi\left(\sum_{i=1}^{n} t_i x_i\right) \leq \sum_{i=1}^{n} t_i \, \varphi(x_i) \tag{4}$$

Lemma 4 follows easily from induction and the definition of convex function.

Lemma 5.

$$x_i \in I \quad , \quad t_i \in (0,1) \quad \Longrightarrow \quad \varphi\left(\frac{\sum_{i=1}^{n} t_i x_i}{\sum_{i=1}^{n} t_i}\right) \leq \frac{\sum_{i=1}^{n} t_i \, \varphi(x_i)}{\sum_{i=1}^{n} t_i} \, . \tag{5}$$

To see Lemma 5, let $\tilde{t}_i = \dfrac{t_i}{\sum_{j=1}^{n} t_j}$, note that $\sum_{i=1}^{n} \tilde{t}_i = 1$, and then use Lemma 4.

Recall 6. Geometric-Arithmetic Mean Inequality (GAM Inequality) ⟨GM ≤ AM⟩

$$x_i \geq 0 \quad , \quad n \in \mathbb{N} \quad \Longrightarrow \quad \left(\prod_{i=1}^{n} x_i\right)^{1/n} \leq \frac{1}{n} \sum_{i=1}^{n} x_i \, .$$

The GAM Inequality follows directly for the Generalized Geometric-Arithmetic Mean Inequality.

Proposition 7 (Generalized Geometric-Arithmetic Mean Inequality)**.**

$$x_i \geq 0 \quad , \quad t_i \in (0,1) \quad , \quad \sum_{i=1}^{n} t_i = 1 \quad \Longrightarrow \quad \prod_{i=1}^{n} x_i^{t_i} \leq \sum_{i=1}^{n} t_i \, x_i \, . \tag{6}$$

The proof of Prop. 7 is part of your homework. See towards the end of this file for further deatils.

Proposition 8 (Young's Inequality)**.** *Let* $1 < p < \infty$.

$$a_i \geq 0 \quad , \quad 1 < p < \infty \qquad \Longrightarrow \qquad a_1 \ \cdot a_2 \ \leq \ \frac{(a_1)^p}{p} \ + \ \frac{(a_2)^{p'}}{p'} \ . \tag{7}$$

The proof of Prop. 8 is part of your homework. See towards the end of this file for further deatils.

Theorem 9 (Hölder's Inequality in ℓ_p)**.** *Let* $1 < p < \infty$.
For sequences $\{x_i\}_{i=1}^N$ *and* $\{y_i\}_{i=1}^N$ *from* $\mathbb{R}$,

$$\left\|\{x_i \cdot y_i\}_{i=1}^N\right\|_{\ell_1} \ \leq \ \left\|\{x_i\}_{i=1}^N\right\|_{\ell_p} \quad \cdot \quad \left\|\{y_i\}_{i=1}^N\right\|_{\ell_{p'}} \tag{8}$$

that is

$$\sum_{i=1}^N |x_i y_i| \ \leq \ \left[\sum_{i=1}^N |x_i|^p\right]^{1/p} \ \cdot \ \left[\sum_{i=1}^N |y_i|^{p'}\right]^{1/p'} \ . \tag{9}$$

Note that when $p = 2 = p'$, Hölder's inequality is just the Cauchy-Schwarz inequality. The proof of Theorem 9 is part of your homework. See towards the end of this file for further deatils.

Theorem 10 (Minkowski's Inequality in ℓ_p)**.** *Let* $1 < p < \infty$.
For sequences $\{x_i\}_{i=1}^N$ *and* $\{y_i\}_{i=1}^N$ *from* $\mathbb{R}$,

$$\left\|\{x_i + y_i\}_{i=1}^N\right\|_{\ell_p} \ \leq \ \left\|\{x_i\}_{i=1}^N\right\|_{\ell_p} \ + \ \left\|\{y_i\}_{i=1}^N\right\|_{\ell_p} \tag{10}$$

that is

$$\left[\sum_{i=1}^N |x_i + y_i|^p\right]^{1/p} \ \leq \ \left[\sum_{i=1}^N |x_i|^p\right]^{1/p} \ + \ \left[\sum_{i=1}^N |y_i|^p\right]^{1/p} \ . \tag{11}$$

The proof of Thm. 10 is part of your homework. See towards the end of this file for further deatils.

Statement of Exercise's Parts a – e.

Metric Space Exercise 2a. Give a really short proof of Proposition 7 (Generalized GAM inequality) by applying Lemma 4 to a cleverly chosen convex function $y = \varphi(x)$. Be sure to carefully justify that your φ is convex.

Proof. Put proof here. □

Metric Space Exercise 2b. Give a really short proof of Proposition 8 (Young's inequality) using the Generalized GAM inequality (Proposition 7).

Proof. Put proof here. □

Metric Space Exercise 2c. Prove Theorem 9 (Hölder's Inequality) using Young's inequality (Proposition 8).

Proof. Put proof here. □

Metric Space Exercise 2d. Prove Theorem 10 (Minkowski's Inequality) using Hölder's inequality (Theorem 9).

Proof. Put proof here. □

Metric Space Exercise 2e. Carefully conclude that $\left(R^N, d_p\right)$ is a metric space when $1 < p < \infty$.

Proof. Put proof here. □