Exercise. Let $R > 0$ and $z_0 \in \mathbb{C}$.

(a) Let $f \in H(B_R(z_0))$ have a zero of order m at z_0. Show $\dfrac{1}{f}$ has a pole of order m at z_0.

(b) Let $f \in H(B'_R(z_0))$ have a pole of order m at z_0. Show $\dfrac{1}{f}$ has a removable singularity at z_0, and furthermore, if we extend $\frac{1}{f}$ to $\widetilde{\frac{1}{f}}$ by defining $\widetilde{\frac{1}{f}}(z_0) = 0$, then $\widetilde{\frac{1}{f}}$ is holomorhic at z_0 and has a zero of order m at z_0.

(c) What is the order of the pole of

$$h(z) := \frac{1}{(2\cos z - 2 + z^2)^2}$$

at $z = 0$? Explain your answer.

Remark. Loosely speaking, this exercise shows that, at $z = z_0$,

(a) f has zero of order $m \Rightarrow \frac{1}{f}$ has pole of order m

(b) f has pole of order $m \Rightarrow \frac{1}{f}$ has zero of order m.