Thm. 2.14. Cauchy's Integral Formula for starlike sets. Let:

1. $\gamma^* \subset G \subset \mathbb{C}$
2. $f \in H(G)$
3. $a \in \mathbb{C} \setminus \gamma^*$

where γ is a closed contour and G is open and starlike. Then

$$[f(a)] \cdot [\mathrm{Ind}_\gamma(a)] = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-a}\, dz \; . \qquad \text{(CIF)}$$

Exercise. Let $\alpha \in \mathbb{C} \setminus \{0\}$ with $|\alpha| \neq 1$. Let $\gamma \colon [0, 2\pi] \to \mathbb{C}$ be given by $\gamma(\theta) := e^{i\theta}$.
(a) Using Cauchy's Integral Formula (CIF) for starlike sets, calculate

$$\int_\gamma \frac{dz}{(z-\alpha)\left(z - \frac{1}{\alpha}\right)}$$

for the case that $|\alpha| < 1$ as well as the case that $|\alpha| > 1$.
When applying CIF for starlike sets, following the notation in the Script of this theorem [Thm. II.2.14, p. 21], clearly indicate what is your: open starlike set G, function $f \colon G \to \mathbb{C}$, and $z_0 \in G \setminus \gamma^*$.
(b) Using part (a), calculate

$$\int_0^{2\pi} \frac{d\theta}{1 - 2\alpha\cos\theta + \alpha^2}$$

for the case that $|\alpha| < 1$ as well as the case that $|\alpha| > 1$. Hint. $e^{i\theta} + e^{-i\theta} = 2\cos\theta$.

Solution Box. Put your final solution in the boxes provided below and then show your justification your solutions below this Solution Box.
(a)

$$\int_\gamma \frac{dz}{(z-\alpha)\left(z - \frac{1}{\alpha}\right)} = \begin{cases} \boxed{\text{a solution goes here}} & \text{if } |\alpha| < 1 \\ \boxed{\text{another solution goes here}} & \text{if } |\alpha| > 1 \; . \end{cases}$$

(b)

$$\int_0^{2\pi} \frac{d\theta}{1 - 2\alpha\cos\theta + \alpha^2} = \begin{cases} \boxed{\text{yet another solutions goes here}} & \text{if } |\alpha| < 1 \\ \boxed{\text{last solution goes here}} & \text{if } |\alpha| > 1 \; . \end{cases}$$

Justification/Proof of the Solutions: