The Summation by Parts Formula. Let $\{a_n\}_{n=1}^{N}$ and $\{b_n\}_{n=1}^{N}$ be finite sequences of complex numbers. Then for $N > M > 1$

$$\sum_{n=M}^{N} a_n b_n \;=\; a_N B_N \;-\; a_M B_{M-1} \;-\; \sum_{n=M}^{N-1} (a_{n+1} - a_n)\, B_n \qquad \text{where} \qquad B_k = \sum_{l=1}^{k} b_l\,. \tag{1}$$

Hint for proof of summation by parts formula: substitute $b_n = B_n - B_{n-1}$ in the sum on the left. You may use, without proving, the *Summation by Parts Forumula*.

Exercise. Prove the power series $\displaystyle\sum_{n=1}^{\infty} \frac{z^n}{n}$ converges for each $z \in \mathbb{C}$ with $|z| = 1$ except $z = 1$.

Hint. Use the above summation by parts. Note if $z \in \mathbb{C} \setminus \{1\}$ with $|z| = 1$, then for each $k \in \mathbb{N}$

$$\left|\sum_{n=1}^{k} z^n\right| \;=\; \left|\frac{z - z^{k+1}}{1 - z}\right| \;\le\; \frac{|z| + \left|z^{k+1}\right|}{|1 - z|} \;=\; \frac{2}{|1 - z|}\,.$$