Exercise. Define $f\colon \mathbb{C} \to \mathbb{C}$ and $u, v\colon \mathbb{R}^2 \to \mathbb{R}$ by

$$\begin{aligned} f(z) &:= \sqrt{|xy|} \qquad \text{where } x := \operatorname{Re} z \quad \text{and} \quad y := \operatorname{Im} z \\ u(x,y) &= \operatorname{Re} f(x+iy) \\ v(x,y) &= \operatorname{Im} f(x+iy) \ . \end{aligned}$$

Show that

1. u and v satisfies the Cauchy Riemann equations at $(x, y) = (0, 0)$
2. f is not differentiable at $z = 0$.

Recall that (complex) differentiable means differentiable as defined in Class Script, p. 4, Def. I.3.1.

Remark. As usual in such a setting, we write $z = x+iy$ with $x, y \in \mathbb{R}$ and let $u(x, y) = \operatorname{Re}(f(x, y))$ and $v(x, y) = \operatorname{Im}(f(x, y))$. Thus $f(x + iy) = u(x, y) + iv(x, y)$.

Recall. The Cauchy-Riemann Equations (CReq) for f are

$$u_x = v_y \qquad \text{and} \qquad u_y = -v_x \ . \tag{CReq}$$

Recall some Big Theorems

Let $f\colon G \to \mathbb{C}$ where G is an open subset of $\mathbb{C}$. Fix $z_0 \in G$.

▷ Diff⇒CR. Let f is differentiable at z_0. Then the CR equations for f are satisfied at z_0.

▷ CR+more⇒Diff. Let the CR equations for f be satisfied at z_0 and, furthermore, the first partial derivatives u_x, v_y, u_y, and v_x:

1. exist in some neighborhood $N_\varepsilon(z_0)$ of z_0
2. be continuous at z_0.

Then f is differentiable at z_0.

▷ If f is differentiable at $z_0 = x_0 + iy_0$, then $f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0) = v_y(x_0, y_0) - iu_y(x_0, y_0)$.
