p33 ## § 3.3 The Residue Thm.

1

Def Let $f \in H(G \setminus \{a\})$ (open) have Laurent expansion abt. $z=a$

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n \quad , \text{ valid } z \in B_R(a).$$

Then residue of f at a denoted $\operatorname{Res}(f,a) :\overset{def}{=} c_{-1}$,

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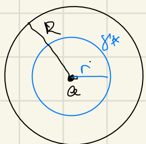
Note Let $f \in H(G \setminus \{a\})$ (open) have Laurent expansion abt. $z=a$

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n \quad , \text{ valid } z \in B_R(a).$$

Let $\gamma_r(t) := a + re^{it}$, $0 \le t \le 2\pi$, $0 < r < R$ and $\gamma^* \subset B_R(a)$



So $\displaystyle\sum_{k=-n}^{n} c_k (z-a)^k \xrightarrow[\longrightarrow]{n\to\infty} f$ on γ^*.

So by uniform convergence ⟨Cor II. 2.6, p17⟩, get

$$\boxed{\sum_{k=-n}^{n} c_k \boxed{\int_{\gamma_r} (z-a)^k \, dz}} \xrightarrow[\text{conv. in } \mathbb{C}]{n\to\infty} \int_{\gamma_r} f(z)\, dz.$$

$$\parallel$$

$$c_{-1}(2\pi i)$$

Ex II. 2.7 p17

$$\boxed{\int_{\gamma_r} (z-a)^k \, dz} = \begin{cases} 0 & , \text{ if } k \ne -1 \\ 2\pi i & , \text{ if } k = -1 \end{cases}$$

So $$\boxed{\operatorname{Res}(f,a) = \frac{1}{2\pi i} \int_{\gamma_r} f(z)\, dz}$$

← This formula is helpful when do not have LSE of f @ a.

If, also, f has a pole of order m at a ⟨so $f(z) = \sum_{k=-m}^{\infty} c_k (z-a)^k$⟩, then

$$\boxed{\operatorname{Res}(f,a) = \lim_{z\to a} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left[(z-a)^m f(z) \right]}$$

2nd diff. the power series term by term (m−1)−times ⟨producing the (m−1)!⟩ & then take $\lim_{z\to a}$ of the (m−1)-times diff. power series

$$= \sum_{k=0}^{\infty} c_{k-m} (z-a)^k \quad \leftarrow \text{1st note}$$

↳ WTF $c_{-1} \overset{note}{=} c_{-m+(m-1)}$

Useful Fact Let $p, q \in H(G)$ (open G) and $z_0 \in G$. <WTF $Res(\frac{p}{q}, z_0)$> $\cancel{\mathbb{Z}}$

If $q(z_0) = 0$ but $p(z_0) \neq 0$ and $q'(z_0) \neq 0$,

then at z_0, $\frac{p}{q}$ has a simple pole (def a pole of order 1) and

$$\boxed{Res\left(\frac{p}{q}, z_0\right) = \frac{p(z_0)}{q'(z_0)}}.$$

Why true?

LTGBG. <handwriting warning $q \neq g$>

(1.) By considering the power series of $q \in H(G)$ at z_0, since $q(z_0) = 0$ & $q'(z_0) \neq 0$,

$$q(z) = (z - z_0)\, g(z)$$

for some $g \in H(G)$ s.t $g(z_0) \neq 0$. <so q has zero @ z_0 of multiplicity 1>

(2.) So $\frac{p(z)}{q(z)} = \frac{p(z)/g(z)}{(z - z_0)}$ and $\frac{p}{g} \in H(G)$ w/ $\frac{p(z_0)}{g(z_0)} \neq 0$.

so $\frac{p}{q}$ has a simple pole at z_0. (ie pole order $m=1$)

(3.) So

$$Res\left(\frac{p}{q}, z_0\right) \overset{\text{w/ } m=1}{=} \lim_{z \to z_0} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left[(z - z_0)^m \frac{p(z)}{q(z)} \right]$$

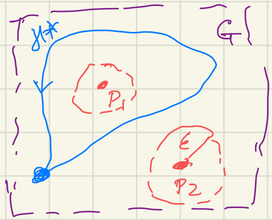
$$\overset{\text{i.e}}{=} \lim_{z \to z_0} (z - z_0)^1 \frac{p(z)}{q(z)} \overset{\text{by (2)}}{=} \lim_{z \to z_0} \frac{p(z)}{g(z)}$$

$$\overset{p, g \in C(G)}{=} \frac{p(z_0)}{g(z_0)} = \frac{p(z_0)}{q'(z_0)}$$

(1) $\Rightarrow q'(z) = g(z) + (z - z_0)\, g'(z)$

$\Rightarrow q'(z_0) = g(z_0) + 0$

p34 Residue Thm (Thm III. 3.1)



Let G be a starlike region (open & connected)
$p_1, \dots, p_n \in G$ distinct points
$f \in H(G \setminus \{p_1, \dots, p_n\})$
γ be a closed piecewise smooth curve (path) in $G \setminus \{p_1, \dots, p_n\}$,

Then

$$\boxed{\int_\gamma f(z)\,dz = 2\pi i \sum_{k=1}^{n} [\operatorname{Res}(f, p_k)]\, \operatorname{Ind}_\gamma(p_k)} .$$

Pf LTGBG. Find $0 < \varepsilon < R$ s.t. for each $k \in \{1, \dots, n\}$:

(1) The Laurent Expansion of f abt p_k
$$f(z) := \sum_{n=-\infty}^{\infty} c_n^{(k)} (z - p_k)^n \text{ is valid in } B_R'(p_k)$$
($G \cup$ note)

(2) The balls $\{B_R(p_k)\}_{k=1}^{n}$ are disjoint.

(3) $\gamma^* \subset \left[\bigcup_{k=1}^{n} B_\varepsilon(p_k)\right]^c \overset{\text{i.e.}}{=} \bigcap_{k=1}^{\infty} [B_\varepsilon(p_k)]^c \overset{\text{i.e.}}{=} \{z \in \mathbb{C} \mid |z - p_k| \geq \varepsilon \quad \forall k = 1, \dots, n\}$

So for each $k = 1, \dots, n$, the singular part of the LE of f abt p_k, ie
$$S^{(k)} := \sum_{n=-\infty}^{-1} c_n^{(k)} (z - p_k)^n \text{ conv. uniformly on } [B_\varepsilon(p_k)]^c .$$

Define $g : G \setminus \{p_1, \dots, p_n\} \to \mathbb{C}$ by:
$$g(z) := f(z) - \sum_{k=1}^{n} S^{(k)}(z) .$$

Note $g \in H(G \setminus \{p_1, \dots, p_n\})$ <b/c f & $S^{(k)}$'s are>.

Note For each k, $g \in H(B_R'(p_k))$

since the balls $\{B_R'(p_k)\}_{k=1}^{n}$ are disjoint (by (2))

Claim: Fix $k \in \{1, .., n\}$. Then $g \in H(B'(p_k))$ extends to $\hat{g} \in H(B(p_k))$.

$\forall\ z \in B_R'(p_k)$

$$g(z) := f(z) - \sum_{j=1}^{n} S^{(j)}(z)$$

$$= \sum_{n=-\infty}^{\infty} c_n^{(k)} (z-p_k)^n - S^{(k)}(z) - \sum_{j\neq k} S^{(j)}(z)$$

$$= \boxed{\underbrace{\sum_{n=0}^{\infty} c_n^{(k)} (z-p_k)^n}_{\in H(B_R(p_k))} - \underbrace{\sum_{j\neq k} S^{(j)}(z)}_{\in H(B_R(p_k)) \text{ b/c } S_j \in H(\mathbb{C}\setminus\{p_j\}) \text{ and } p_j \notin B_R(p_k)}}$$

So what's in the purple box defines a holom. (thus continuous) function on $B_R(p_k)$.

So $\lim_{z\to p_k} g(z)$ exists, set $\hat{g}(p_k) := \lim_{z\to p_k} g(z)$.

And p_k is a remov. sing. of g on $B_R(p_k)$ and $\hat{g} \in H(B_R(p_k))$.

Thus g extends from $G\setminus\{p_1, \dots, p_n\}$ to $\hat{g} \in H(G)$.

Cauchy's Thm for ☆-like sets (Thm II.2.12 p20) $\Rightarrow$ $\int_\gamma g(z)\,dz = 0$

$$\Rightarrow \int_\gamma f(z)\,dz = \sum_{k=1}^{n} \int_\gamma S^{(k)}(z)\,dz \qquad (4)$$

Fix k.
$$\sum_{n=-j}^{-1} c_n^{(k)} (z-p_k)^n \xrightarrow[\quad]{j\to\infty} S^{(k)}(z) \quad \text{on } \gamma^*$$
$$\Downarrow$$
$$\sum_{n=-j}^{-1} c_n^{(k)} \left[\int_\gamma (z-p_k)^n\,dz\right] \xrightarrow[\text{in } \mathbb{C}]{j\to\infty} \int_\gamma S^{(k)}(z)\,dz$$

$$\begin{cases} \text{if } n\neq -1: (z-p_k)^n = \dfrac{d}{dz}\dfrac{(z-p_k)^{n+1}}{n+1} \xRightarrow[\text{for paths}]{\text{FTC}} \int_\gamma (z-p_k)^n\,dz = 0 \\ \text{if } n = 1: \operatorname{Ind}_\gamma(p_k) :\overset{\text{def}}{=} \dfrac{1}{2\pi i}\int_\gamma \dfrac{1}{z-p_k}\,dz. \end{cases}$$

So $\int_\gamma S^{(k)}(z)\,dz = c_{-1}^{(k)} (2\pi i) \operatorname{Ind}_\gamma(p_k)$. $\qquad (5)$

So $\int_\gamma f(z)\,dz \overset{\text{by}}{\underset{(4)\ \&\ (5)}{=}} \sum_{k=1}^{n} \operatorname{Res}(f, p_k)\,(2\pi i) \operatorname{Ind}_\gamma(p_k)$. ☒

<u>Def</u> Let $f : \mathbb{R} \Rightarrow \mathbb{R}$ be st $f|_{[-R,R]}$ is Riemann integrable $\forall R > 0$.

The <u>Cauchy principle value integral of f</u> is:

$$(P.V) \int_{-\infty}^{\infty} f(x)\,dx \;:=\; \lim_{R \to \infty} \int_{-R}^{R} f(x)\,dx \,;$$

provided the limit exists.

<u>Note</u>

(1) $\int_{-\infty}^{\infty} x\,dx$ DNE but $(PV) \int_{-\infty}^{\infty} x\,dx = 0$.

(2) f even function $\Rightarrow \int_{-R}^{R} f(x)\,dx = 2 \int_{0}^{R} f(x)\,dx,$

Ex (Typical Qual Question) : Find the principle value of

$$\int_{-\infty}^{\infty} \frac{x^2}{x^6+1}\,dx$$

using the Residue Theorem.

Consider $\boxed{f(z) = \dfrac{z^2}{z^6+1}}$.

$z^6 = -1 = e^{i\pi} = e^{i(\pi + 2\pi k)}$, $k \in \mathbb{Z}$

$f(z) \in H(\mathbb{C} \setminus \{p_0, p_1, \ldots, p_5\})$ where p_k's are the "6th roots of -1",

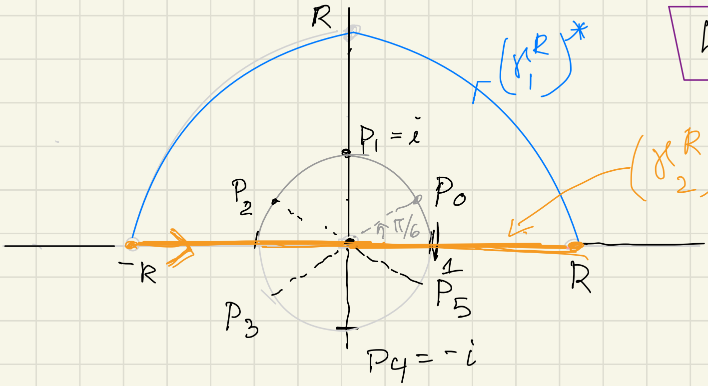
star like

ie.

$$p_k = e^{i\left(\frac{\pi}{6} + \frac{2\pi k}{6}\right)} \quad \text{for } k = 0, 1, \ldots, 5.$$

To apply Res. Thm.. need closed path that doesn't pass thru the p_k's (ie pw smooth)
(and also want to some how pick up $[-R, R] \subset \mathbb{R}$.

$\boxed{\text{Let } R > 1}$



$\gamma_1^R : [0, \pi] \to \mathbb{C}$ w/ $\gamma_1^R(t) = Re^{it}$

$\gamma_2^R : [-R, R] \to \mathbb{C}$ w/ $\gamma_2^R(t) = Re^{it}$

γ^R is the join of γ_1^R and γ_2^R = "semicircle"

$\boxed{\int_{-R}^{R} f(x)\,dx}$

$\parallel$

Idea $\boxed{\int_{\gamma^R} f(z)\,dz} = \boxed{\int_{\gamma_1^R} f(z)\,dz} + \boxed{\int_{\gamma_2^R} f(z)\,dz}$

|| use Res. Thm, w/ R big enuf. some "nice" number

as $R \to \infty$ hope $\to 0$

0

PV $\int_{-\infty}^{\infty} f(x)\,dx$.

Apply Res. Thm w/ $f(z) = \dfrac{z^2}{z^6+1}$

$$\int_{\Gamma R} f(z)\,dz = 2\pi i \sum_{k=0}^{2} \left[\mathrm{Res}(f, p_k)\right] \cancel{\mathrm{Ind}_{\Gamma^R}(p_k)} \to 1$$

Let's compute $\mathrm{Res}(f, p_k)$ for $k = 0, 1, 2$.

$$\mathrm{Res}\left(\frac{z^2}{z^6+1}, p_k\right) = \text{"}\frac{p(p_k)}{q'(p_k)}\text{"} = \frac{p_k^2}{6p_k^5} = \frac{1}{6}p_k^{-3}$$

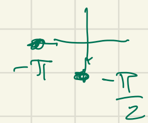
Useful Fact Let $p, q \in H(G)$ (open) and $z_0 \in G$. <WTF $\mathrm{Res}(\frac{p}{q}, z_0)$>

If $q(z_0) = 0$ but $p(z_0) \neq 0$ and $q'(z_0) \neq 0$, then at z_0, $\frac{p}{q}$ has a simple pole (def a pole of order 1) and

$$\boxed{\mathrm{Res}\left(\frac{p}{q}, z_0\right) = \frac{p(z_0)}{q'(z_0)}}.$$

← Apply w/
$p(z) = z^2$
$q(z) = z^6 + 1$
$z_0 = p_k$.

$$= \frac{1}{6}\left(e^{i\frac{\pi}{6}} e^{\frac{i\pi k}{3}}\right)^{-3} = \frac{1}{6} e^{-i\frac{\pi}{2}} \left(e^{-i\pi}\right)^k$$

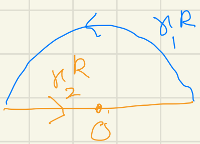


$$= \frac{1}{6}(-i)(-1)^k = \boxed{(-1)^{k+1}\left(\frac{i}{6}\right)}$$

So, when $R > 1$,

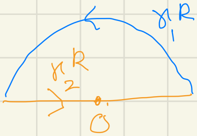
$$\int_{\Gamma^R} f(z)\,dz = 2\pi i \left[\sum_{k=0}^{2} (-1)^{k+1}\left(\frac{i}{6}\right)\right] = (2\pi i)\left(\frac{i}{6}\right)(-1+1-1) = \frac{\pi}{3}.$$

Next try to show $\left|\int_{\gamma_1^R} f(z)\,dz\right| \xrightarrow{R\to\infty} 0$.



Often the ML (max/length) Lemma is helpful.
<Prop 2.5 (3) p 16>. Need $f \in C(\gamma^*)$ w/ γ path.

$$\left|\int_\gamma f(z)\,dz\right| \leq \left[\max_{z\in\gamma^*} |f(z)|\right] \ell(\gamma)$$

Recall 

and $\gamma_1^R(t) = Re^{it}$, $0 \le t \le \pi$.

By the ML lemma

$$\left|\int_{\gamma_1^R}\frac{z^2}{z^6+1}\,dz\right| \le \ell(\gamma_1^R)\left[\max_{z\in(\gamma_1^R)^*}\left|\frac{z^2}{z^6+1}\right|\right]$$

$$\le (\pi R)\left[\max_{z\in(\gamma_1^R)^*}\frac{|z|^2}{|z|^6-1}\right] = \pi R\left[\frac{R^2}{R^6-1}\right] = \frac{\pi R^3}{R^6-1} \xrightarrow{R\to\infty} 0.$$

if $R>1$ ... use reverse Δ ineq

So

$$\int_{\Gamma R}\frac{z^2}{z^6+1}\,dz = \int_{\gamma_1^R}\frac{z^2}{z^6+1}\,dz + \int_{\gamma_2^R}\frac{z^2}{z^6+1}\,dz$$

$\int_{\Gamma R}\frac{z^2}{z^6+1}\,dz = \frac{\pi}{3}$ when $R>1$; $\int_{\gamma_1^R}\frac{z^2}{z^6+1}\,dz \to 0$ as $R\to\infty$; $\int_{\gamma_2^R}\frac{z^2}{z^6+1}\,dz = \int_{-R}^{R}\frac{x^2}{x^6+1}\,dx$.

So

$$PV\int_{-\infty}^{\infty}\frac{x^2}{x^6+1}\,dx \overset{\text{def}}{=} \lim_{R\to\infty}\int_{-R}^{R}\frac{x^2}{x^6+1}\,dx = \boxed{\frac{\pi}{3}},$$

Ex. Find $\int_0^\infty \frac{x^2}{x^6+1}\,dx$.

note $f(x) := \frac{x^2}{x^4+1}$ is even

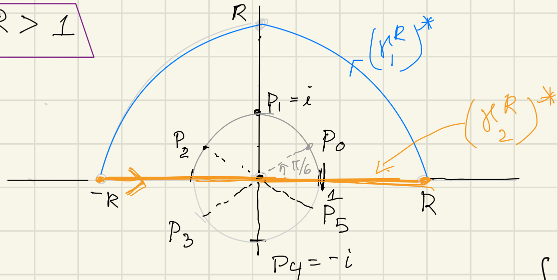
$$\int_0^\infty\frac{x^2}{x^6+1}\,dx = \lim_{R\to\infty}\int_0^R\frac{x^2}{x^6+1}\,dx = \lim_{R\to\infty}\frac{1}{2}\int_{-R}^{R}\frac{x^2}{x^6+1}\,dx$$

$$\overset{\text{last Ex.}}{=} \frac{1}{2}\left(\frac{\pi}{3}\right) = \frac{\pi}{6}.$$

Ex Revisit. Find PV $\int_{-\infty}^{\infty} \frac{x^2}{x^6+1}\,dx$.

- Had $f(z) = \frac{z^2}{z^6+1} \in H(\mathbb{C} \setminus \{p_0, p_1, \dots, p_5\})$ w/ $p_k = e^{i\left(\frac{\pi}{6} + \frac{2\pi k}{6}\right)}$ i.e. p
- To apply Res. Thm.. need closed path (i.e. piecewise smooth curve) that doesn't pass thru the p_k's (and also want to some how pick up $[-R, R] \subset \mathbb{R}$.

Let $R > 1$

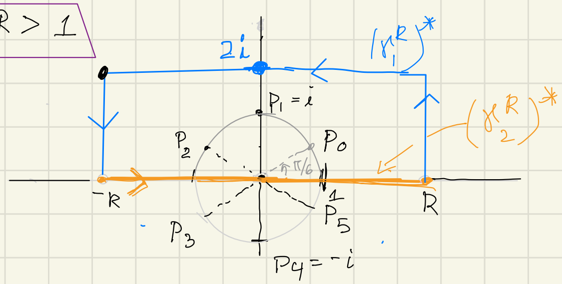


Then it was not too hard to show $\int_{\gamma_1^R} f(z)\,dz \xrightarrow{R\to\infty} 0$.

↳ this is often the case when f is a rational function and γ_1^R is a semicircle.

- But sometime, when trying to find a closed path that doesn't pass thru the pts. where f is not differentiabe, a nice semicircle circle will not work, Another commonly used path to try is:

Let $R > 1$



But you need to also make sure that

$$\int_{\gamma_1^R} f(z)\,dz \xrightarrow{R\to\infty} 0,$$

[this "rectangle" is not a good choice for $f(z) = \frac{z^2}{z^6+1}$.