§3.1 Zeros of holomorphic functions

Defs Let $G \subset \mathbb{C}$.

1. $G \subseteq \mathbb{C}$ is a <u>region</u> provided G is open and connected
2. z_0 is a <u>zero</u> of a function $f: G \to \mathbb{C}$ provided $f(z_0) = 0$.
3. For $f: G \to \mathbb{C}$, the <u>zero set of f</u> is $Z_f := \{ z \in G \mid f(z) = 0 \}$

Defs Let X be metric space and $A \subset X$.

- A' = set of limit pts of A. $x_0 \in A' \Leftrightarrow \forall \varepsilon > 0, \ N'_\varepsilon(x_0) \cap A \neq \emptyset$.
- I_A = set of isolated pts of A. $x_0 \in I_A \Leftrightarrow \exists \varepsilon > 0$ st $N_\varepsilon(x_0) \cap A = \{x_0\}$.

So: $I_A = A \setminus A' \subset A$.

Thm 3.1.1 Let: $G \subset \mathbb{C}$ be a region and $G \neq \emptyset$.

$f \in H(G)$

I_{Z_f} := the set of isolated points of Z_f. <note $I_{Z_f} \subseteq Z_f$>

Then the following hold.

(i) Either $Z_f = G$ or $Z_f = I_{Z_f}$.

(ii) $\forall\, a \in I_{Z_f}$

$\exists!\ m \in \mathbb{N}$ <m is called the <u>order of the zero</u> $a \in Z_f$>

$\exists\ g \in H(G)$ s.t. $g(a) \neq 0$ and

$$f(z) = (z-a)^m g(z) \qquad \forall z \in G$$

(iii) If $Z_f \neq G$, then Z_f is (at most) countable.

Lemma $f \in H(G)$ with G open $\Rightarrow Z_f'$ is both open and closed in G.

Pf Ash Lemma 2.4.7 - read (nice use of power series)

Pf of (i) Equiv. to Show:

(1i) $Z_f = G \Rightarrow Z_f \neq I_{Z_f}$

(2i) $Z_f \neq G \Rightarrow Z_f = I_{Z_f}$

(1i) Let $Z_f = G$. $\langle$WTS $Z_f \neq I_{Z_f}\rangle$.

Fix $z_0 \in Z_f \stackrel{\text{note}}{=} G \leftarrow$ open.

Claim $z_0 \notin I_{Z_f}$

$\exists\, \varepsilon_{z_0} > 0$ st if $0 < \varepsilon < \varepsilon_{z_0}$ then $N_\varepsilon(z_0) \subset G$
$\Downarrow$
$N_\varepsilon(z_0) \subset Z_f$
$\Downarrow$
$N_\varepsilon(z_0) \cap Z_f = N_\varepsilon(z_0) \neq \{z_0\}$

So $I_{Z_f} = \emptyset \neq G = Z_f$.

(2i) Let $Z_f \neq G$. $\langle$WTS $Z_f = I_{Z_f}$. Since $I_{Z_f} \subset Z_f$, WTS $Z_f \subset I_{Z_f}$.$\rangle$

WLOG, $Z_f \neq \emptyset$. Fix $a \in Z_f$. $\langle$WTS $a \in I_{Z_f}\rangle$

1. Find $r > 0$ st $B_r(a) \subset G$
2. $f \in H(G) \xRightarrow{\text{Thm 2.23}} \exists!\, \{c_n\}_{n=1}^{\infty} \subset \mathbb{C}$ st $f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n \;\; \forall z \in B_r(a)$
3. $a \in Z_f \Rightarrow c_0 = 0$
4. Claim $\exists\, n \in \mathbb{N}$ st $c_n \neq 0$.

If $c_n = 0 \;\forall n \in \mathbb{N}$, then $B_r(a) \subset Z_f$
$\Downarrow$
$\emptyset \neq \overline{B_r(a)} = [B_r(a)]' \subset Z_f'$ $\leftarrow$ by Lemma open & closed in connected set G.

Since the only clopen subset of a connected set G are G and $\emptyset$, $Z_f' = G$. So

$G = Z_f' \underset{\substack{\uparrow \\ \text{by cont. of } f}}{\subset} Z_f \Rightarrow G = Z_f$ $\rightarrow\leftarrow$

5. Set $m := \min\{n \in \mathbb{N} : c_n \neq 0\}$

< So on $B_r(a)$

$$f(z) = \sum_{\underline{n=m}}^{\infty} c_n (z-a)^n = (z-a)^m \sum_{n=m}^{\infty} c_n (z-a)^{n-m} = (z-a)^m \underbrace{\sum_{n=0}^{\infty} c_{n+m} (z-a)^n}_{\text{our candidate for } g \text{ on } B_r(a)}$$

but want g on all of G so... >

6. Define $g: G \to \mathbb{C}$ by

$$g(z) = \begin{cases} \dfrac{f(z)}{(z-a)^m} & \text{if } z \in G \setminus \{a\} \\ c_m & \text{if } z = a, \end{cases}$$

7. $\underline{\text{Clearly}}$ $f(z) = (z-a)^m g(z) \quad \forall z \in G$

[Case $z \neq a$: by design
Case $z = a$: $f(a) \underset{\underset{a \in Z_f}{\uparrow}}{=} 0 = (a-a)^m g(a)$.

8. $\underline{\text{Clearly}}$ $g \in H(G \setminus \{a\})$ < since $f \in H(G \setminus \{a\})$ >

9. Have $f(z) = (z-a)^m g(z) \quad \forall z \in G$

and $g(z) = \underbrace{\sum_{n=0}^{\infty} c_{n+m} (z-a)^n}_{\text{has Rad. of conv } \geq r} \quad \forall z \in B_r(a)$

$$\xrightarrow[\substack{\text{power series abt "a" are} \\ \text{holm. on } B_R(a) \\ R \uparrow \text{ rad. of conv.}}]{\text{Thm I-4.7}} g \in H(B_r(a)) \xrightarrow[g \in H(G \setminus \{a\})]{f \in H(G)} g \in H(G).$$

10. So $g \in C(B_r(a))$ and $g(a) = c_m \neq 0 \Rightarrow \exists r_1 \in (0, r)$ s.t.

$\Rightarrow$ if $z \in B_{r_1}(a)$ then $g(z) \neq 0$

$\Rightarrow$ if $z \in B'_{r_1}(a)$ then $f(z) = (z-a)^m g(z)$

$\Rightarrow$ if $z \in B'_{r_1}(a)$ then $f(z) \neq 0$.

$\Rightarrow$ $B'_{r_1}(a) \cap Z_f = \{a\}$

$\boxed{\exists}$ to (i)

11. $a \in I_{Z_f}$.

(ii) Did in Proof of (i)

(iii) Let $Z_f \neq G$.

$\forall a \in Z_f \quad \exists r_a > 0$ st $\underbrace{B_{r_a}(a)}_{\substack{\ni \\ z_a \in \mathbb{Q} \times i\mathbb{Q}}} \cap Z_f = \{a\}$.

The map $\begin{array}{ccc} Z_f & \longrightarrow & \mathbb{Q} \times i\mathbb{Q} \\ a & \longmapsto & z_a \end{array}$ is 1-to-1.

Recall from metric spaces.

Let $A \subset \mathbb{C}$

$A' :=$ the set of limit points of A

Then $a \in A' \overset{\text{def}}{\iff} \forall \varepsilon > 0, \; N'_\varepsilon(a) \cap A \neq \emptyset$

$\overset{\text{prop}}{\iff} \exists \{a_n\} \subset A \setminus \{a\}$ s.t. $a_n \to a$.

WLOG, distinct

Cor Identity Thm

Let • $f, g \in H(G)$ ← region <open & connected> in $\mathbb{C}$

• $f|_S = g|_S$ for some $S \subseteq G$

• $S' \cap G \neq \emptyset$.

Then $f(z) = g(z) \quad \underline{\forall z \in G}$

Pf

Consider $h: G \to \mathbb{C}$ defined by $h(z) = f(z) - g(z)$.

So $\boxed{S \subset Z_h}$ < WTS $Z_h = G$ so E_{Aut} TS $Z_h \neq I_{Z_h}$ (Prin. of Isolated zeros) < going to show $s_0 \in Z_h$ but $s_0 \notin I_{Z_h}$ >

Pick $s_0 \in S' \cap G$.

$\Downarrow$

$\exists \{s_n\}_n \subset S \setminus \{s_0\}$ s.t. $s_n \to s_0$

$\Downarrow_n$ h is cont.

$0 = h(s_n) \to h(s_0)$

$\uparrow$

$s_n \in S \subset Z_h$

$\Downarrow$

$h(s_0) = 0$

$\Downarrow$

$s_0 \in Z_h$

But $\underset{\substack{\text{m} \\ S \,\subset\, Z_h}}{s_n} \to s_0 \Rightarrow s_0 \notin I_{Z_h}$. ∎

~~Do~~ the Example & uniqueness of extension.

Why the identity thm Rocks. How to def $\cos z$ & e^z for $z \in \mathbb{C}$? | 3.6

① Ash's book § 2.3 (Complex Trig. Function)

Def.

$$\sin z := \frac{e^{iz} - e^{-iz}}{2i} \qquad \boxed{\cos z := \frac{e^{iz} + e^{-iz}}{2}}$$

def $e^{i\theta} = \cos\theta + i\sin\theta$

$\boxed{x \in \mathbb{R}}$

$$\frac{e^{ix} + e^{-ix}}{2} = \frac{(\cos x + \cancel{i\sin x}) + (\cos(-x) + \cancel{i\sin(-x)})}{2}$$

$$e^{ix} = e^{i} = \cos x$$

start w/ $\tilde{g} : \mathbb{R} \to \mathbb{R}$ $\tilde{g}(x) = \cos x$,

extend (if can) $g : \mathbb{C} \to \mathbb{C}$ st $g \in H(\mathbb{C})$

$g|_{\mathbb{R}} = \tilde{g}$.

② $e^z = e^{x+iy} := e^x \cos y + i e^x \sin y$.

$x \in \mathbb{R}$ $\boxed{e^{x+i0} = e^x}$