Recall: **Cor I.4.8** (script p11/notes p 28)

Let $\quad f(z) = \sum_{n=0}^{\infty} C_n (z-a)^n \quad$ (PS)

have radius of conv. $R \neq 0$.

Then $\forall k \in \mathbb{N}$ and $z \in B_R(a) = \{z \in \mathbb{C} : |z-a| < R\}$

(1) $f^{(k)}(z)$ exists

(2) $f^{(k)} \in H(B_R(a))$

(3) $f^{(k)}(z) = \sum_{n=k}^{\infty} C_n \, n(n-1) \dots (n-k+1) \, (z-a)^{n-k}$

(4) $c_k = \dfrac{f^{(k)}(a)}{k!}$ (so c_k's are unique)

Thm 2.23 (Power series expansion of holomorphic functions)

Let: $a \in B_R(a) \subset G \overset{\text{open}}{\subset} \mathbb{C}$

$f \in H(G)$.

Then $\exists \, ! \, \{C_n\}_{n=0}^{\infty} \subset \mathbb{C}$ st

$$f(z) = \sum_{n=0}^{\infty} C_n (z-a)^n$$

$\forall z \in B_R(a)$.

Recall Cauchy's Integral Formula (for ☆-like sets)

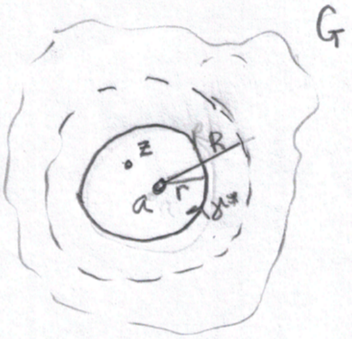
Let (a) $\gamma^* \subset G \subset \mathbb{C}$ (b) $f \in H(G)$ (c) $z \in G \setminus \gamma^*$

↳ open & ☆-like

↳ γ is closed contour

Then $[f(z)] \, [\mathrm{Ind}_\gamma(z)] = \dfrac{1}{2\pi i} \displaystyle\int_\gamma \dfrac{f(w)}{w-z} \, dw. \quad$ (CIF)

__Let's think__ How to use CIF to express f from Thm 2.23 as an integral?



__Pf__ __Claim 1__ $\exists$ such c_n's.

- Let $0 < r < R$.
- Fix $z \in B_r(a)$.
- Def $\gamma : [0, 2\pi] \to B_R(a)$ by $\gamma(t) = a + re^{it}$.

- $z \in B_r(a) \implies \operatorname{Ind}_\gamma(z) = 1$.

- CIF (applied to set $B_R(a) = G$) $\implies \forall z \in B_r(a)$

$$f(z) = \frac{1}{2\pi i} \int_\gamma \boxed{\frac{1}{w-z}} f(w)\, dw$$

- For $w \in \gamma^*$, $\left|\frac{z-a}{w-a}\right| = \frac{|z-a|}{r} \overset{z \in B_r(a)}{<} 1$, and so

$$\frac{1}{w-z} \overset{A}{=} \frac{1}{w-a}\left[\frac{1}{1 - \frac{z-a}{w-a}}\right] \overset{GS}{=} \lim_{N\to\infty} \underbrace{\frac{1}{w-a} \sum_{n=0}^{N} \frac{(z-a)^n}{(w-a)^{n+1}}}_{:= g_N(w)}$$

so $g_N : \gamma^* \to \mathbb{C}$

Will need to say $\int \Sigma = \Sigma \int$ so want $\rightrightarrows$.
and futhermore, (for this fixed $z \in B_r(a)$)

$$g_N(\cdot) \overset{\text{on } \gamma^*}{\rightrightarrows} \frac{1}{(\cdot) - z}.$$

- So $f(z) \underset{GS}{\overset{CIF}{=}} \frac{1}{2\pi i} \int_\gamma f(w) \sum_{n=0}^{\infty} \frac{(z-a)^n}{(w-a)^{n+1}}\, dw$

$$\underset{\text{Cor 2.6}}{\overset{UC}{=}} \sum_{n=0}^{\infty} \underbrace{\left[\frac{1}{2\pi i} \int_\gamma \frac{f(w)}{(w-a)^{n+1}}\, dw\right]}_{:= c_n} (z-a)^n \quad ✓$$

$\leftarrow$ $\boxed{\text{will use in pf. of next Cor.}}$

- __Claim 2__ uniqueness. Follows from Ch 1 Cor 4.8, which implies

If $H(z) := \sum_{n=0}^{\infty} c_n (z-a)^n$ has nonzero Rad. of conv, then $c_n = \frac{f^{(n)}(a)}{n!}$.

Cor 2.24a CIF (yet another version, this one for derivatives)

Let $G \overset{open}{\subseteq} \mathbb{C}$

$f \in H(G)$

$n \in \mathbb{N} \cup \{0\}$.

don't recall what happened to (2).

Then (1) $f^{(n)} \in H(G)$.

If furthermore, $B_R(a) \subset G$, then

$$(3) \quad \boxed{f^{(n)}(a) = \frac{n!}{2\pi i} \int_{\gamma_r} \frac{f(w)}{(w-a)^{n+1}}\, dw}$$

where $\gamma_r : [0, 2\pi] \to \mathbb{C}$

$\gamma_r(t) = a + re^{it}$

$0 < r < R$

$$(4) \quad |f^{(n)}(a)| \le \frac{n!}{R^n} \left[\sup_{z \in B_R(a)} |f(z)| \right] \quad \left(\text{Cauchy's Estimate} \right)$$

Pf LTGBG.

(1) Follows from proof of Ch2 Thm 2.23 ⊕ Ch 1 Cor 4.8 (see page 56)

(3) We can write $f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$ & we now have 2 forumulae for c_n's

$$\frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{(w-a)^{n+1}} \; \underset{\text{Thm II.2.23}}{\overset{\text{pf of}}{=}} \; c_n \overset{\text{cor I.4.8}}{=} \frac{f^{(n)}(a)}{n!}$$

(4) Let $0 < r < R$. Then $\max_{w \in \gamma_r^*} |f(w)| \le \sup_{w \in B_R(a)} |f(w)| = M$

$$(3) \Rightarrow |f^{(n)}(a)| \overset{ML}{\le} \frac{n!}{2\pi} \underbrace{\frac{M}{r^{n+1}}}_{max} \underbrace{(2\pi r)}_{\ell(\gamma_r)} = \frac{n!}{r^n} M \xrightarrow{r \nearrow R} \frac{n!}{R^n} M$$

Cor 2.24b CIF (yet another version).

Let $B_R(a) \subset G \overset{open}{\subseteq} \mathbb{C}$

$f \in H(G)$

$n \in \mathbb{N} \cup \{0\}$.

Then

Thm II.2.23 → ① $\exists! \{c_n\} \subset \mathbb{C}$ st $f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n \quad \forall z \in B_R(a)$

Cor I.4.8 → ② $f^{(n)}(a) = (n!)\, c_n$

Motivation Cauchy's Thm for triangles (Thm II.2.10)

$$\left[f \in H(\underset{\text{open} \subseteq \mathbb{C}}{G}) \right] \Rightarrow \left[\int_{\partial \Delta} f(z)\, dz = 0 \ , \ \forall \text{ triangles } \Delta \subset G \right]$$

Then we used Cauchy's Thm for Δ to get

Cauchy's Thm for $\star$-like sets (Thm II.2.12):

$$\left[\begin{array}{l} f \in H(\underset{\text{open} \subset \mathbb{C} \text{ \& } \star\text{-like}}{G} \setminus \{p\}) \\ f \text{ cont. @ } p \end{array} \right] \Rightarrow \left[f = F' \text{ on } G \text{ w/ } F(z) := \int_{[a, z]} f(w)\, dw \right]$$

(a: star center)

Thm 2.25 Morera's Thm

Let $f : G \to \mathbb{C}$ be continuous (G open $\subseteq \mathbb{C}$)

$\int_{\partial \Delta} f(z)\, dz = 0 \quad \forall$ triangles $\Delta \subset G$. $(*_\Delta)$

Then $f \in H(G)$.

Sketch of Proof LTGBG. Fix $B_R(a) \subset G$ w/ $R > 0$. ETS $f \in H(B_R(a))$.

Define $F : B_R(a) \to \mathbb{C}$ by $F(z) := \int_{[a, z]} f(w)\, dw$.

Proceed as in our proof of Cauchy's Thm for $\star$-like sets (Thm II.2.12), using now $(*_\Delta)$ instead of Cauchy's Thm for Δ, show that: $\boxed{F' = f \text{ on } B_R(a)}$.

$F \in H(B_R(a))$ and

$\Downarrow$ Cor II.2.24

$\boxed{F' \in H(B_R(a))} \Rightarrow f \in H(B_R(a))$

$\hookrightarrow$ Recall says $g \in H(G) \xRightarrow{\forall n \in \mathbb{N}} g^{(n)} \in H(G)$.

Thm 2.26 Liouville's Thm "A bounded entire function is constant."

Let $f \in H(\mathbb{C})$ and $\sup_{z \in \mathbb{C}} |f(z)| := M < \infty$.

Then f is a constant function <ie $\exists\, c \in \mathbb{C}$ st $f(z) = c$ $\forall z \in \mathbb{C}$>
[in fact $c = Me^{i\theta}$ $\exists\, \theta \in \mathbb{R}$.

Pf LTGBG, Let

$$f(z) = \sum_{n=0}^{\infty} c_n z^n \qquad (ps)$$

be the power series expansion of f about zero
<which we know exists by Thm 2.23 (power series expansion of Holom. fns.)>.

- $f \in H(\mathbb{C}) \Rightarrow$ radius of convergence of (ps) is ∞.
- So ETS $c_n = 0$ $\forall n \geq 1$ <for then $f(z) = c_0$ $\forall z \in \mathbb{C}$>
- Know $c_n = \dfrac{f^{(n)}(0)}{n!}$
- Fix $n \in \mathbb{N} = \{1, 2, \dots\}$. For $R > 0$

$$|c_n| = \frac{1}{n!}\left[|f^{(n)}(0)|\right]$$

$\downarrow$ Cauchy's Estimate (Cor 2.24a)

$$\leq \frac{1}{n!}\left[\frac{n!}{R^n}\left(\sup_{z \in B_R(0)} |f(z)|\right)\right]$$

$$\overset{(CH)}{\leq} \frac{M}{R^n} \xrightarrow{R \to \infty} 0.$$

Application of Liouville's Theorem.

(Yet another proof of the FTA).

FTA Let $p(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree $n \geq 1$.

Then $\exists\, z_0 \in \mathbb{C}$ such that $p(z_0) = 0$.

Pf. . LTGBG.

- Assume $\forall z \in \mathbb{C}$, $p(z) \neq 0$. <WTF $\rightarrow\leftarrow$>.
- Then we can define $g : \mathbb{C} \to \mathbb{C}$ by $g(z) = \frac{1}{p(z)}$.

Claim 1: $g \in H(\mathbb{C})$ ✓ <so g is entire>

Claim 2 g is bounded on $\mathbb{C}$;

- Pick $R > 0$ (so big) s.t.

$$\boxed{\sum_{j=0}^{n-1} \frac{|a_j|}{R^{n-j}} < \frac{|a_n|}{2}} \quad (2.0)$$

So $\forall z \notin B_R(0)$. <so for z with $|z|$ sufficiently big

$$(2.1) \quad \left| \sum_{j=0}^{n-1} \frac{a_j}{z^{n-j}} \right| \overset{\Delta}{\leq} \sum_{j=0}^{n-1} \frac{|a_n|}{|z|^{n-j}} \overset{|z| \geq R}{\leq} \sum_{j=0}^{n-1} \frac{|a_j|}{R^{n-j}} \underset{(2.1)}{<} \frac{|a_n|}{2}.$$

So $\forall z \notin B_R(0)$ <next page>

$$|p(z)| = \left| z^n \left[a_n + \frac{a_{n-1}}{z} + \dots + \frac{a_0}{z^n} \right] \right|$$

$$= |z|^n \left| a_n + \frac{a_{n-1}}{z} + \dots + \frac{a_0}{z^n} \right|$$

$$\overset{\Delta}{\geq} |z|^n \left[|a_n| - \underbrace{\left| \sum_{j=0}^{n-1} \frac{a_j}{z^{n-j}} \right|}_{(2.1)} \right]$$

$$\geq |z|^n \left[|a_n| - \frac{|a_n|}{2} \right]$$

$$= |z|^n \frac{|a_n|}{2} \geq \frac{R^n |a_n|}{2} \overset{\text{note}}{>} 0$$

So

(2.2) $$\sup_{z \notin B_R(0)} |q(z)| \leq \frac{2}{R^n |a_n|}.$$

But q is continuous on the compact set $\overline{B_R(0)}$ so

$$\sup_{z \in \overline{B_R(0)}} |q(z)| < \infty.$$

So q is bnd. $\mathbb{C}$ <so claim 2 holds>.

<u>Claim 3</u> Liouville to the rescue

q is a bounded entire function $\xRightarrow{\text{Liouville}}$

q is constant $\Rightarrow$ p is constant. $\Rightarrow$

$\deg p = 0$ $\rightarrow\leftarrow$.

So, yes, yet another proof of FTA is accomplished. $\boxed{\text{Finish Ch 2.}}$