Goal If $a \in \mathbb{C} \setminus \gamma^*$ for a closed contour γ, then

$$\text{Ind}_\gamma(a) := \frac{1}{2\pi i} \int_\gamma \frac{\boxed{1}}{z-a}\, dz \qquad (*)$$

Next we will replace the constant function $\boxed{1}$ in $(*)$ by a "nice" function f and will get

$$[f(a)] \cdot [\text{Ind}_\gamma(a)] \overset{\text{goal}}{=} \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-a}\, dz.$$

Big Time Important Thm

Thm 2.14 Cauchy's Integral Formula (for starlike sets)

Let (a) $\gamma^* \subset G \subset \mathbb{C}$
↳ open & starlike
↳ γ is a closed contour (ie. closed piecewise smooth curve)

(b) $f \in H(G)$

(c) $a \in G \setminus \gamma^*$.

Then

$$\boxed{[f(a)] \cdot [\text{Ind}_\gamma(a)] = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-a}\, dz.} \qquad (CIF)$$

Rmk Pretty nifty.... we can calculate $f(a)$ by a contour integral and the value of $f(a)$ depends only on the values of f on the path going "around but not through a"
< and the $\text{ind}_\gamma(a)$ if wind around & around & around >

Pf. LTGBG. T.F.A.E.;

$$f(a)\left[\mathrm{Ind}_\gamma(a)\right] = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a}\,dz \qquad \text{(CIF)}$$

$\downarrow$ Def 2.13

$$\frac{1}{2\pi i}\int_\gamma \frac{f(a)}{z-a}\,dz = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a}\,dz$$

$$\frac{1}{2\pi i}\int_\gamma \frac{f(z)-f(a)}{z-a}\,dz = 0 \qquad \text{(WTS)}.$$

Define $g: G \to \mathbb{C}$ by (using fact $f \in H(G)$)

$$g(z) = \begin{cases} \dfrac{f(z)-f(a)}{z-a} & z \in G\setminus\{a\} \\ f'(a) & z = a. \end{cases} \xrightarrow{\text{note}} \begin{bmatrix} g \in H(G\setminus\{a\}) \\ g \text{ continuous on } G. \end{bmatrix}$$

So g satisfies the conditions of Cauchy's ie theorm. for starlike sets (i.e. Thm 2.12). So by Thm 2.12

$$\int_\gamma g(z)\,dz = 0.$$

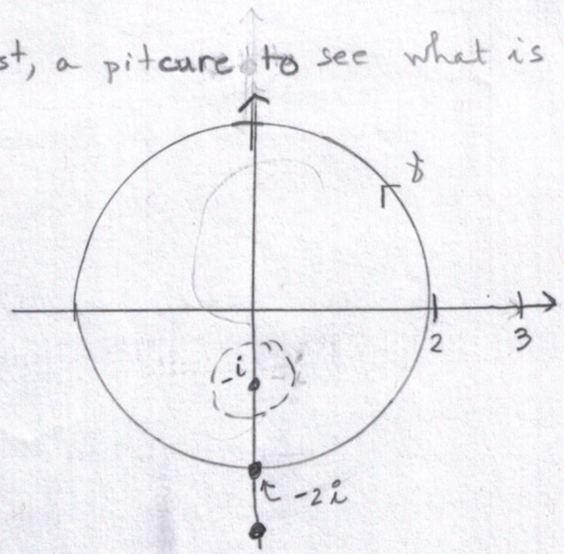
So (WTS) indeed holds. ∎

Example Let $\gamma : [0, 2\pi] \to \mathbb{C}$ be given by $\gamma(t) = 2e^{it}$. Find

$$\int_\gamma \frac{z}{(9-z^2)(z+i)}\, dz .$$

Note $\dfrac{z}{(9-z^2)(z+i)} = \dfrac{z}{(3-z)(3+z)(z+i)}$

Sol'n First, a pitcure to see what is going on.



So define $f : \mathbb{C} \setminus \{\pm 3\} \to \mathbb{C}$ by

$$f(z) = \frac{z}{(3-z)(3+z)}$$

So

$$\frac{z}{(9-z^2)(z+i)} = \frac{f(z)}{z-(-i)} .$$

Let $G = \{ z \in \mathbb{C} \mid |z| < 2.17 \}$.

So $f \in H(G)$ and $-i \in G \setminus \gamma^*$.

So by Cauchy's integral formula (for starlike domains, Thm 2.14)

$$\int_\gamma \frac{f(z)}{z+i}\, dz = [2\pi i] \cdot [\operatorname{Ind}_\gamma(-i)]\,[f(-i)]$$

$\downarrow$ Winding # Lemma

$$= [2\pi i] \quad 1 \quad \frac{-i}{9-(-i)^2}$$

$$= 2\pi i \left[\frac{-i}{10}\right] = \boxed{\frac{\pi}{5}}$$

Thm 2.16 Fundamental Theorem of Algebra

Can prove using Cauchy's Integral Formula (Thm. 2.14)

See course script.

Next Let's "get a handle" on winding number.

Prop 2.17 Let: (a) γ be a closed contour

(b) $a \in \mathbb{C} \setminus \gamma^*$.

Then $\mathrm{Ind}_\gamma(a) \in \mathbb{Z}$.

Pf LTGBG, say $\gamma : [b,c] \to \mathbb{C}$. Let $\overset{\text{"bad set"}}{B} := \{t \in [b,c] : \gamma'(t) \text{ DNE or is not cont.}\}$

Since γ is a closed piecewise smooth curve, B is a finite set. Note:

$$\mathrm{Ind}_\gamma(a) \overset{def}{=} \frac{1}{2\pi i}\int_\gamma \frac{dz}{z-a} \overset{def}{=} \frac{1}{2\pi i}\int_b^c \frac{\gamma'(s)}{\gamma(s)-a}\,ds.$$

So let's explore $g : [b,c] \to \mathbb{C}$ where

$$g(t) := \int_b^t \frac{\gamma'(s)}{\gamma(s)-a}\,ds.$$

Note:

(1) $g(b) = 0$

(2) g is continuous on $[b,c]$

(3) $g'(t) = \dfrac{\gamma'(t)}{\gamma(t)-a} \quad \forall\, t \in (b,c) \setminus B$.

Next consider $h : [b,c] \to \mathbb{C}$ where

$$h(t) := e^{-g(t)}\,(\gamma(t)-a).$$

Note:

(4) h is continuous on $[b,c]$

(5) $\forall\ t \in [b,c] \setminus B$

$$\frac{d}{dt} h(t) = \frac{d}{dt}\left[e^{-g(t)} (\gamma(t) - a) \right]$$

$$= e^{-g(t)} \gamma'(t) - g'(t)\, e^{-g(t)} (\gamma(t) - a)$$

$$= e^{-g(t)} \left[\gamma'(t) - g'(t)(\gamma(t) - a) \right] \overset{(3)}{=} 0.$$

So h is constant on $[b,c]$. Since

$$h(b) = e^{-g(b)} (\gamma(b) - a) \overset{(1)}{=} \gamma(b) - a$$

and

$$h(c) = e^{-g(c)} (\gamma(c) - a) \overset{\gamma \text{ closed}}{=\!=} e^{-g(c)} (\gamma(b) - a),$$

$$e^{-g(c)} = 1 \quad \text{and so} \quad g(c) \in \{ 2\pi i m \mid m \in \mathbb{Z} \}.$$

$$\text{Ind}_\gamma(a) = \frac{1}{2\pi i}\, g(c) \in \mathbb{Z}.$$

$\blacksquare$

<u>Prop 2.18</u> Let: $C_0 \subseteq G \subseteq \mathbb{C}$

G — open

C_0 — connected component of G.

Then:

(1) C_0 is open (in $\mathbb{C}$)

(2) the relative closure (in G) of C_0 is connected;

$\quad = \bigcap \{ F \cap G : C_0 \subseteq F \cap G \text{ and } F \text{ is closed in } \mathbb{C} \}$.

(3) $G = \bigcup_{j \in \Gamma} C_j$ (disjoint; Γ a finite or countable set) where each C_j is:

(a) a connected component of G

(b) open in $\mathbb{C}$

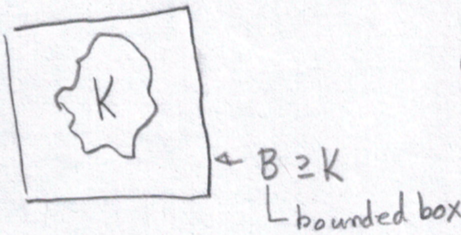
(c) relatively closed in G

Pf Did earlier this semester.

Remark 2.19 • If $K \subseteq \mathbb{C}$ is compact, then $\mathbb{C} \setminus K$ has exactly one unbounded component.

• If γ is a closed contour, <then $\gamma^* \subset \mathbb{C}$ is compact and so> then $\mathbb{C} \setminus \gamma^*$ has exactly 1 unbounded component.

Idea



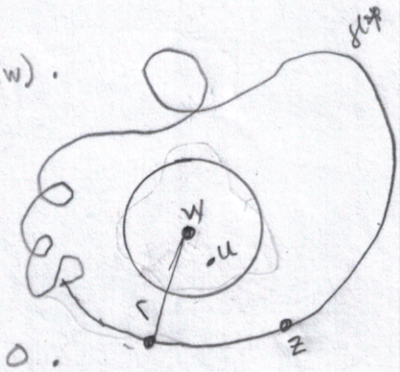
$\mathbb{C} \setminus B$ (which is out here) is clearly path connected $\Downarrow$ connected

Thm 2.20 Let γ be a closed contour. Then: <here, component = connected component>

(1) $\mathrm{Ind}_\gamma : \mathbb{C} \setminus \gamma^* \to \mathbb{Z}$ is constant on each component of $\mathbb{C} \setminus \gamma^*$

(2) $\mathrm{Ind}_\gamma(a) = 0 \quad \forall\, a$ in the unbounded component of $\mathbb{C} \setminus \gamma^*$

Proof. LTGBG. 1 Define $f : \mathbb{C} \setminus \gamma^* \to \mathbb{C}$ by $f(w) = \mathrm{Ind}_\gamma(w)$.



Claim 1 f is continuous.

Fix $w \in \mathbb{C} \setminus \gamma^*$.

γ^* compact $\Rightarrow r := d(w, \gamma^*) > 0$.

Fix $\varepsilon > 0$. Let $\delta \overset{(1)}{:=} \min\left\{ \frac{r}{2}, \frac{\varepsilon \pi r^2}{\ell(\gamma)} \right\} \overset{\text{note}}{>} 0$.

Fix $u \overset{(2)}{\in} N_\delta(w)$. Note for $z \in \gamma^*$:

(3) $|z - w| \geq \inf_{z_1 \in \gamma^*} |z_1 - w| =: d(w, \gamma^*) =: r$

(4) $|z - u| \overset{\triangle}{\geq} \underset{\downarrow (3)}{|z - w|} - \underset{\downarrow (2)}{|w - u|}$

$> r - \delta \geq r - \frac{r}{2} = \frac{r}{2}$.

Thus

$$|f(w) - f(u)| = \frac{1}{2\pi} \left| \int_\gamma \frac{(w-u)}{(z-w)(z-u)}\, dz \right| \overset{ML}{\leq} \frac{\ell(\gamma)}{2\pi} \max_{z \in \gamma^*} \frac{|w-u|}{|z-w| \cdot |z-u|}$$

$$\overset{(2)(3)(4)}{<} \frac{\ell(\gamma)}{2\pi} \frac{\delta}{r^2/2} = \frac{\delta \ell(\gamma)}{\pi r^2} \overset{(1)}{\leq} \varepsilon.$$

Claim 2 f is constant on each (connected) component of $\mathbb{C}\setminus\gamma^*$.

Let C be a component of $\mathbb{C}\setminus\gamma^*$.

f continuous $\Rightarrow f(C)$ is a connected subset of $\mathbb{C}$.

But $f(C) \overset{\text{Thm } 2.20}{\subseteq} \mathbb{Z}$. So $f(C)$ consists of a single point

Claim 3 $\operatorname{Ind}_\gamma(a) = 0 \quad \forall a \in C_0 :=$ the unbounded component of $\mathbb{C}\setminus\gamma^*$.

- Find $R_1 > 0$ st $\gamma^* \subset \overline{B_{R_1}(0)}$ <can do b/c γ^* is compact>
- Note $\left[\overline{B_{R_1}(0)}\right]^c \subset C_0$

  $\underbrace{\left[\overline{B_{R_1}(0)}\right]^c}_{\text{(path) connected}} \subset \mathbb{C}\setminus\gamma^* = \bigcup_{i\in\Pi} C_i$ w/ C_i components of $\mathbb{C}\setminus\gamma^*$

  $\Rightarrow \exists\, i \in \Pi$ st $\underbrace{\left[\overline{B_{R_1}(0)}\right]^c}_{\text{unb.}} \subset C_i$

  So $C_i = C_0$

- Find $R_2 > R_1$ so that, $\forall z \in \gamma^*$, $|R_2 - z| > \dfrac{\ell(\gamma)}{\pi}$.

  For any $z \in \gamma^*$ and $R > 0$ $\qquad z \in \gamma^* \subset \overline{B_{R_1}(0)}$

  $|R - z| \overset{\Delta}{\geq} |R - R_1| - |z - R_1| \geq |R - R_1| - 2R_1 \xrightarrow{R\to\infty} \infty$.

- So
$$|\operatorname{Ind}_\gamma(R_2)| = \frac{1}{2\pi}\left|\int_\gamma \frac{dz}{z - R_2}\right| \overset{ML}{\leq} \frac{\ell(\gamma)}{2\pi} \max_{z\in\gamma^*} \frac{1}{|z - R_2|}$$
$$\leq \frac{\ell(\gamma)}{2\pi} \cdot \frac{\pi}{\ell(\gamma)} = \frac{1}{2}$$

  But $\operatorname{Ind}_\gamma : \mathbb{C}\setminus\gamma^* \to \mathbb{Z}$ so $\operatorname{Ind}_\gamma(R_2) = 0$.

  Since $\operatorname{Ind}_\gamma$ is constant on C_0, $\operatorname{Ind}_\gamma(a) = 0 \quad \forall a \in C_0$

Example 2.21 &
Prop. 2.22] read

Recall Cor I.4.8 (Script p11 / notes p 28)

Let $$f(z) = \sum_{n=0}^{\infty} C_n (z-a)^n \qquad \text{(PS)}$$

have radius of conv. $R \neq 0$.

Then $\forall\, k \in \mathbb{N}$ and $z \in B_R(a) = \{z \in \mathbb{C} : |z-a| < R\}$

(1) $f^{(k)}(z)$ exists

(2) $f^{(k)} \in H(B_R(a))$

(3) $f^{(k)}(z) = C_n \; n(n-1) \dots (n-k+1) \; (z-a)^{n-k}$

(4) $C_k = \dfrac{f^{(k)}(a)}{k!}$ (so C_k's are unique)

Thm 2.23 (Power series expansion of holomorphic functions)

Let $a \in B_R(a) \subset G \overset{\text{open}}{\subset} \mathbb{C}$

$f \in H(G)$.

Then $\exists\, ! \;\{C_n\}_{n=0}^{\infty} \subset \mathbb{C}$ st

$$f(z) = \sum_{n=0}^{\infty} C_n (z-a)^n$$

$\forall\, z \in B_R(a)$.

Recall Cauchy's Integral Formula (for ☆-like sets)

Let (a) $\gamma^* \subset G \subset \mathbb{C}$ (b) $f \in H(G)$

↳ open & ☆-like   (c) $z \in G \setminus \gamma^*$

↳ γ is closed contour

Then $$[f(z)]\,[\mathrm{Ind}_\gamma(z)] = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w-z}\, dw. \qquad \text{(CIF)}$$