Cor 2.6 Let:

- $\gamma : [a,b] \to \mathbb{C}$ be a piecewise smooth curve
- $f_n : \gamma^* \to \mathbb{C}$ continuous $\forall n \in \mathbb{N}$
- $f : \gamma^* \to \mathbb{C}$ a function
- $f_n \rightrightarrows f$ on γ^* (i.e. f_n conv. uniformly to f on γ^*).

Then $\int_\gamma f_n(z)\,dz \to \int_\gamma f(z)\,dz$ (i.e. $\lim_{n\to\infty} \left| \int_\gamma f_n(z)\,dz - \int_\gamma f(z)\,dz \right| = 0$)

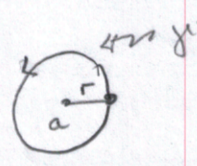
Pf See Script. Key calculation:

$$\left|\int_\gamma f_n(z)\,dz - \int_\gamma f(z)\,dz\right| \underset{\text{Prop 2.5 (5)}}{=} \left|\int_\gamma (f_n(z)-f(z))\,dz\right| \underset{\text{Prop 2.5 (3)}}{\overset{ML}{\leq}} \overbrace{\left[\max_{z\in\gamma^*} |f_n(z)-f(z)|\right]}^{\xrightarrow{n\to\infty} 0 \text{ by unif. conv. assuption}} \ell(\gamma).$$

Important

Ex 2.7. Let $\gamma : [0, 2\pi] \to \mathbb{C}$ be given by $\gamma(t) = \overset{\mathbb{C}}{\underset{\omega}{a}} + \overset{0}{\hat{r}}e^{it}$. Then

$$\frac{1}{2\pi i}\int_\gamma \frac{1}{(z-a)^n}\,dz = \begin{cases} 1, & n = 1 \\ 0, & n \in \mathbb{Z}\setminus\{1\}. \end{cases}$$



The Calculation: <think $z = \gamma(t)$>

$$\int_\gamma (z-a)^{-n}\,dz = \int_0^{2\pi} (\gamma(t)-a)^{-n}\gamma'(t)\,dt = \int_0^{2\pi} (re^{it})^{-n}(ire^{it})\,dt = ir^{1-n}\int_0^{2\pi} e^{it(1-n)}\,dt$$

$$\begin{cases} \underline{\underline{n\neq 1}} \quad ir^{1-n} \left.\dfrac{e^{it(1-n)}}{i(1-n)}\right|_0^{2\pi} = \dfrac{ir^{1-n}}{i(1-n)}\left(e^{i2\pi(1-n)} - e^0\right) = \dfrac{r^{1-n}}{1-n}(1-1) = 0 \\ \underline{\underline{n=1}} \quad ir^{1-1}\displaystyle\int_0^{2\pi} 1\,dt = i\,2\pi. \end{cases}$$

Note

- The integral is independent of the radius r.
- The next theorem will allow us to extend this simple example (for $n \neq 1$) to arbitrary closed contours γ st $a \notin \gamma^*$.

A "FTC" for path integrals.

Thm 2.8 Let:

(1) $\gamma : [a,b] \to \mathbb{C}$ be a piecewise smooth curve, i.e. a path / contour

(2) F be holomorphic on γ^*, i.e. ⟨by Def I.3.1⟩ F be holomorphic on an open set containing γ^*

(3) F' be continuous on γ^*.

Then

(4) $$\int_\gamma F'(z)\,dz = F(\gamma(b)) - F(\gamma(a)).$$

And so, if γ is closed, then

$$\int_\gamma F'(z)\,dz = 0.$$

In Ex 2.7 we had

$$F'(z) = \frac{1}{(z-a)^n} \text{ w/ } \gamma^* \text{ (circle around } a)$$

$$\Downarrow n \neq 1$$

$$F(z) = \frac{1}{(-n+1)} \frac{1}{(z-a)^{n-1}}$$

for Cor 2.9 ↓ to

Pf LTGBG.

Case 1: γ is smooth.

$$\int_\gamma F'(z)\,dz \overset{\text{def of path integral}}{=} \int_a^b F'(\gamma(t))\,\gamma'(t)\,dt \overset{\text{CR}}{=} \int_a^b (F\circ\gamma)'(t)\,dt \overset{\mathbb{R}\text{-variable FTC}}{=} (F\circ\gamma)(b) - (F\circ\gamma)(a)$$

Case 2: γ is piecewise smooth.

Break into smooth pieces. (See Script for details).

Cor 2.9 Let γ be a closed contour and $a \in \mathbb{C}$. Then

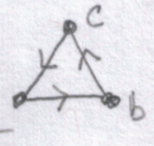
(1) $\int_\gamma (z-a)^n\,dz = 0$ if $n \in \mathbb{N} \cup \{0\}$

(2) $\int_\gamma \frac{1}{(z-a)^n}\,dz = 0$ if $n \in \mathbb{N}\setminus\{1\}$ and $a \notin \gamma^*$

Compare to Ex 2.7, where γ^* is (circle around a)

Pf Apply FTC!

Notation Triangle $\Delta(a,b,c)$ for an ordered triple (a,b,c) of complex numbers



$[a,b]$ denotes the directed line segment (so a curve) from a to b.

$$\partial\Delta(a,b,c) = [a,b] \cup [b,c] \cup [c,a]$$

$$\Delta(a,b,c) = \partial\Delta(a,b,c) \cup \text{"all pts inside"}$$

<u>Thm 2.10</u> (Cauchy's Thm for a triangle)

Let:

(a) $p \in G \overset{\text{open}}{\subseteq} \mathbb{C}$

(b) $f : G \to \mathbb{C}$

(c) $f \in H(G \setminus \{p\})$

(d) f continuous on G.

Side remark: we will see later that such an f is actually in $H(G)$.

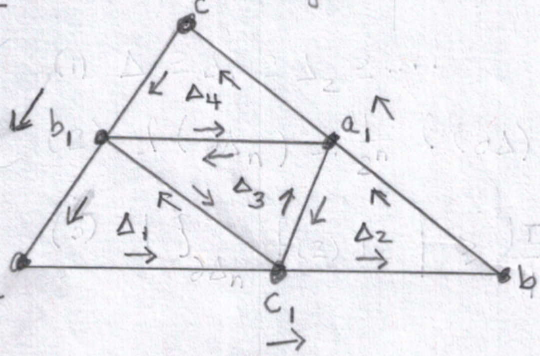
Consider a triangle $\Delta = \Delta(a,b,c) \subset G$. Then

$$\int_{\partial\Delta} f(z)\, dz = 0.$$

<u>Pf.</u> LTGBG. Let $I := \int_{\partial\Delta} f(z)\,dz$. $\langle$WTS $I = 0\rangle$ $\langle \exists$ 3 Cases $\begin{cases} 1.\ p \notin \Delta(a,b,c) \\ 2.\ p \in \{a,b,c\} \\ 3.\ p \in \Delta(a,b,c) \setminus \{a,b,c\} \end{cases}$ $\rangle$

<u>Case 1 : $p \notin \Delta$</u> . Let $L := \ell(\partial\Delta)$.

<u>Step 1.</u> Define triangles $\Delta_1, \Delta_2, \Delta_3, \Delta_4$ as indicated in below picture.



- a_1, b_1, c_1 are midpoints of the line segments upon which they sit
- explain orientation of Δ_j s.

$\langle$used $\int_\gamma f(z)dz = -\int_{-\gamma} f(z)dz$ (Prop 2.5)$\rangle$

Then $I = \sum_{j=1}^{4} \int_{\partial\Delta_j} f(z)\,dz$. So there is at least on j s.t.

$$\left|\int_{\partial\Delta_j} f(z)\,dz\right| \geq \frac{|I|}{4}.$$ WLOG (relabel if needed)

$$\left|\int_{\partial\Delta_1} f(z)\,dz\right| \geq \frac{|I|}{4^1}.$$

Note $\ell(\partial\Delta_1) = \frac{1}{2}\ell(\partial\Delta)$.

Next divide similarly Δ_1 into 4 triangles and find $\Delta_2 \subseteq \Delta_1$ st

$$\left|\int_{\partial\Delta_2} f(z)\,dz\right| \geq \frac{|I|}{4^2} \text{ and } \ell(\partial\Delta_2) = \tfrac{1}{2}\ell(\partial\Delta_1).$$

Continue in this fashion to construct a sequence $\{\Delta_n\}_{n=1}^{\infty}$ of triangles s.t., $\forall\, n \in \mathbb{N}$,

(i) $\Delta_0 \supseteq \Delta_1 \supseteq \Delta_2 \supseteq \ldots$

(ii) $\left| \int_{\partial\Delta_n} f(z)\, dz \right| \geq \frac{|I|}{4^n}$

(iii) $\ell(\partial\Delta_n) = \frac{1}{2^n} \cdot L(\partial\Delta)$.

This finishes step 1.

Since if $\bigcap_{n=1}^{\infty} \Delta_n = \emptyset$, then $\{\Delta_n^c : n \in \mathbb{N}\}$ would be an open covering (of the compact set Δ) without a finite subcovering.

Now fix $\varepsilon > 0$.

f is differentiable at z_0 $\Uparrow$ assumption (C) that $f \in H(G \setminus \{p\})$

(1) $\begin{bmatrix} \Delta \text{ compact} \\ \{\Delta_n\}_{n=1}^{\infty} \text{ has f.i.p.} \end{bmatrix} \Rightarrow \exists\, z_0 \in \bigcap_{n=1}^{\infty} \Delta_n \subseteq \Delta \xrightarrow{\text{case 1}} z_0 \neq p$

f.i.p. = finite intersection property, i.e. $\cap$ of any finite # Δ_n's is nonempty

(2) So $\exists\, r > 0$ s.t. $\forall\, z \in B_r(z_0)$

$$|f(z) - f(z_0) - f'(z_0)(z - z_0)| \leq \varepsilon |z - z_0| .$$

(3) $\begin{bmatrix} z_0 \in \bigcap_{n=1}^{\infty} \Delta_n \\ \ell(\Delta_n) \xrightarrow[\text{(iii)}]{n\to\infty} 0 \end{bmatrix} \Rightarrow \exists\, N \text{ st } \Delta_N \subset B_r(z_0)$

Claim 2 $|I| < \varepsilon L^2$.

Well

$$\frac{|I|}{4^N} \overset{(ii)}{\leq} \left| \int_{\partial\Delta_N} f(z)\, dz \right|$$

$$= \left| \int_{\partial\Delta_N} f(z)\, dz - f(z_0) \overbrace{\int_{\partial\Delta_N} 1\, dz}^{=0 \text{ by Ex 2.4 : } \int_\gamma 1 dz = \gamma(b) - \gamma(a)} - f'(z_0) \underbrace{\int_{\partial\Delta_N} (z - z_0)\, dz}_{=0 \text{ by Thm 2.8 <FTC for path } \int \text{>}} \right|$$

$D_z \underbrace{\left[\tfrac{1}{2}(z-z_0)^2\right]}_{\in H(\partial\Delta_N)} = \underbrace{(z - z_0)}_{\text{cont on } \partial\Delta_N}$

$$= \left| \int_{\partial\Delta_N} \left[f(z) - f(z_0) - f'(z_0)(z-z_0) \right] dz \right|$$

prop 2.5 (3)

$$\underset{ML}{\leq} \ \ell(\partial\Delta_N) \left[\max_{z \in \partial\Delta_N} |f(z) - f(z_0) - f'(z_0)(z-z_0)| \right]$$

$\downarrow$ (iii) $\qquad \downarrow \partial\Delta_N \subset B_r(z_0)$ by (2)

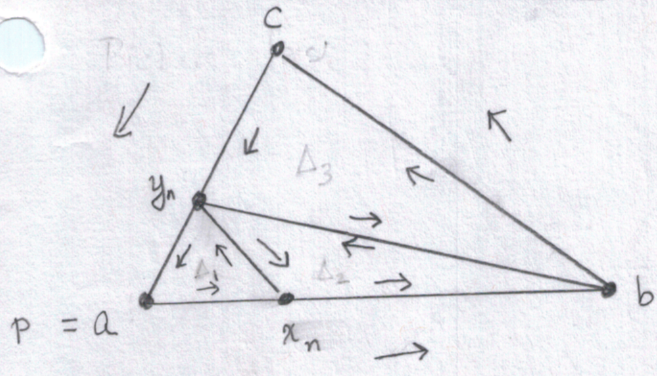
(3)+(2)

$$\leq \ 2^{-N} L \left[\max_{z \in \partial\Delta_N} \varepsilon |z - z_0| \right]$$

$$\leq \ 2^{-N} \cdot L \cdot \varepsilon \cdot \ell(\partial\Delta_N) = 4^{-N} \varepsilon L^2 .$$

So claim 2 holds. So Case 1 holds. <let $\varepsilon \downarrow 0$ so $\varepsilon L^2 \downarrow 0$>

<u>Case 2 : p is a vertex of $\Delta(a,b,c)$.</u> WLOG, $p = a$. Picture first.



Key idea:

$$\int_{\partial\Delta(a,b,c)} f(z)\,dz = \int_{\partial\Delta(a,x_n,y_n)} f(z)\,dz + \int_{\partial\Delta(x_n,b,y_n)} f(z)\,dz + \int_{\partial\Delta(y_n,b,c)} f(z)\,dz$$

and by case 1, • and • are zero.

Pick a seq. $\{x_n\}$ from the interior of the line segment from a to b s.t. $x_n \to a$.

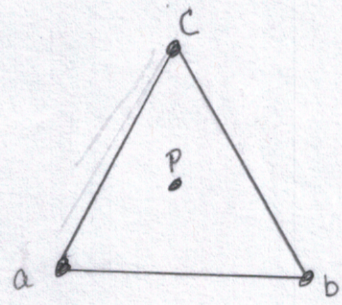
Pick a seq. $\{y_n\}$ " a to c st $y_n \to a$.

Since f is cont. on Δ, $M := \max_{z \in \Delta} |f(z)| < \infty$. So

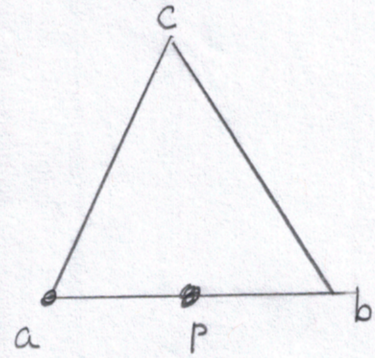
$$\left| \int_{\partial\Delta} f(z)\,dz \right| \underset{\text{idea}}{\overset{\text{Key}}{=}} \left| \int_{\partial\Delta(a,x_n,y_n)} f(z)\,dz \right| \overset{ML}{\leq} M \cdot \ell(\partial\Delta(a,x_n,y_n)) \xrightarrow{n\to\infty} 0 .$$

Case 3 $p \in \Delta(a,b,c) \setminus \{a,b,c\}$. (Key idea)

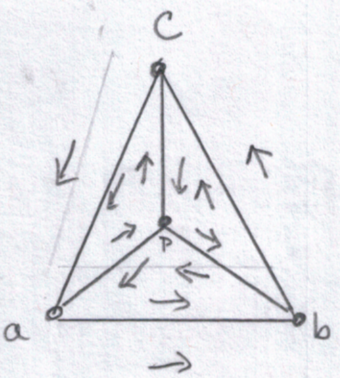
Then one of the following pictures happens.

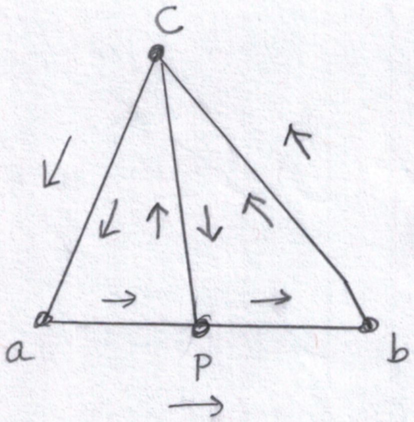


or



reduce to case 2 by





Finally done with Proof of Cauchy's Thm for triangles. ▭

Next we consider more general (than triangles) subsets of $\mathbb{C}$.

he is called a star center of S.

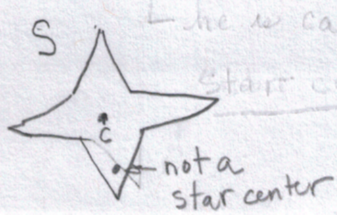
Def 2.11 Let $S, C \subseteq \mathbb{C}$.

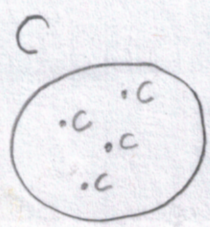
1. S is starlike $\Leftrightarrow$ $\exists\ c \in S$ st $\forall z \in S$, the line segment $[c,z] \subset S$
2. C is convex $\Leftrightarrow$ $\forall\ c \in C$ and $\forall z \in C$, the line segment $[c,z] \subset C$

[starlike ⊕ each pt is star center]

$\Downarrow$

[convex]





Goal: given "nice" $f: S \to \mathbb{C}$, want to def $F: S \to \mathbb{C}$ via $F(z) = \int_{[c,z]} f(w)\,dw$ and hope $F' = f$ on G

Thm 2.12 Cauchy's Thm. for starlike sets

Let :

(a) $p \in G \overset{\text{open}}{\subseteq} \mathbb{C}$ and G be starlike

(b) $f : G \to \mathbb{C}$

(c) $f \in H(G \setminus \{p\})$

(d) f continuous on G.

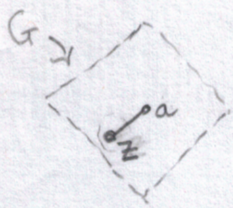
side remark: later we'll see that such f is in $H(G)$.

Then $\exists$ $F \in H(G)$ s.t.

(e) $F' = f$ on G.

(f) $\int_\gamma f(z)\, dz = 0$ for each closed contour γ in G.

Pf. LTGBG. Let $a \in G$ be a star center of. G.



So $\forall z \in G$, the line segment $[a, z] \subset G$.

< Who is a "candidate for F"? >.

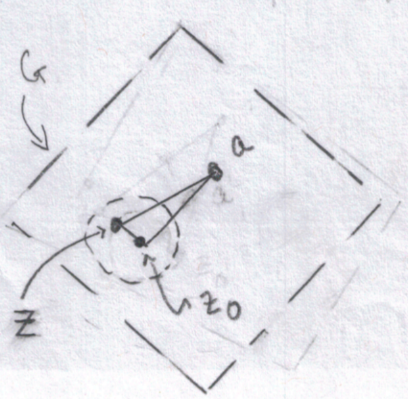
Define $F : G \to \mathbb{C}$ by <can do b/c $[a,z]$ is a smooth path in G & f is cont. on G>

$$F(z) := \int_{[a,z]} f(w)\, dw .$$ < see where using "star like" >

$\boxed{\text{Part e}}$

Fix $z_0 \in G$. <WTS $F'(z_0)$ exists and $F'(z_0) = f(z_0)$.>

Find $r > 0$ s.t. $B_r(z_0) \subset G$. Consider $z \in B_r(z_0)$.



G starlike $\Rightarrow \Delta(a, z_0, z) \subset G$.

So by Cauchy Thm. for triangles :

$$0 = \int_{\Delta(a, z_0, z)} f(w)\, dw$$

$$= \left[\int_{[a,z_0]} + \int_{[z_0,z]} + \int_{[z,a]} \right] f(w)\, dw$$

$$= \left[\int_{[a,z_0]} + \int_{[z_0,z]} - \int_{[a,z]} \right] f(w)\, dw.$$

Thus

$$F(z) - F(z_0) \overset{\text{by def of } F}{:=} \int_{[a,z]} f(w)\,dw - \int_{[a,z_0]} f(w)\,dw = \int_{[z_0,z]} f(w)\,dw.$$

So $\forall\, z \in B_r'(z_0)$

$$\left| \frac{F(z)-F(z_0)}{z-z_0} - f(z_0) \right| \quad \Big\langle \text{used } \int_{[z_0,z]} 1\,dw \overset{\text{Ex 2.4 (or FTC)}}{=} z - z_0 \Big\rangle$$

$$= \left| \frac{1}{z-z_0} \int_{[z_0,z]} \big(f(w) - f(z_0) \big)\, dw \right|$$

$$\overset{ML}{\leq} \frac{1}{|z-z_0|} \left[\max_{w \in [z_0,z]} |f(w) - f(z_0)| \right] \ell([z_0,z])$$

$$= \max_{w \in [z_0,z]} |f(w) - f(z_0)|$$

$$\xrightarrow[f \text{ cont. at } z_0]{z \to z_0} 0 .$$

Thus $F'(z_0) = f(z_0)$ and so (e) holds.

In (e) we showed $F'(z) = f(z)\ \forall\, z \in G$.

So $F \in H(G)$.

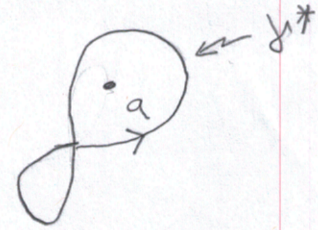
Part (f) $\langle$that $\forall$ closed contours γ in G, $0 \overset{\text{WTS}}{=} \int_\gamma f(z)\,dz \overset{\text{part (e)}}{=} \int_\gamma F'(z)\,dz \rangle$ follows directly from Thm 2.8 $\langle$, the FTC for path integrals$\rangle$.

$\blacksquare$

Def 2.13

Let :

- γ be a <u>closed</u> path ← i.e. piecewise smooth curve
- $a \in \mathbb{C} \setminus \gamma^*$

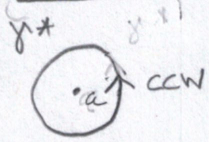


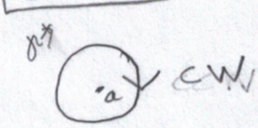
Then the <u>index of γ w.r.t. a</u>, (also called the <u>winding number of γ around a</u>) is

$$\boxed{\operatorname{Ind}_\gamma(a) := \frac{1}{2\pi i} \int_\gamma \frac{dz}{z-a}}$$

<u>Ex</u> Let $\gamma(t) = \overset{\mathbb{C}}{\overset{\in}{a}} + \overset{0}{\hat{r}} e^{ict}$ for $0 \le t \le 2\pi \overset{\mathbb{N}}{\overset{\in}{n}}$ where $c \in \{\pm 1\}$.

$\boxed{c = 1}$ ← "wind" around n-time → $\boxed{c = -1}$





$$\operatorname{Ind}_\gamma(a) := \frac{1}{2\pi i} \int_\gamma \frac{dz}{z-a} = \frac{1}{2\pi i} \int_0^{2\pi n} \frac{1}{re^{ict}} \left(rice^{ict} \right) dt$$

$$\overset{CH}{=} \frac{c}{2\pi} \int_0^{2\pi n} 1 \, dt = \frac{c(2\pi n)}{2\pi}$$

$$= cn$$

↑ # times we wind around curve

orientation : ⊕ for CCW and ⊖ for CW.

Observations about index < why do they mix well w/ previous Ex? >

1) Let γ be a closed contour and $z_0 \in \mathbb{C} \setminus \gamma^*$. Then

$$\mathrm{Ind}_{-\gamma}(z_0) = -\mathrm{Ind}_{\gamma}(z_0).$$

Sketch of Pf.

Let $\gamma : [a,b] \to \mathbb{C}$. So $-\gamma(\bullet) := \gamma(a+b-\bullet) : [a,b] \to \mathbb{C}$. So

$$2\pi i \, \mathrm{Ind}_{-\gamma}(z_0) := \int_{-\gamma} \frac{dz}{z-a} = \int_{t=a}^{t=b} \frac{1}{\gamma(a+b-t)-z_0} \gamma'(a+b-t)(-1)\,dt$$

$$\overset{s=a+b-t}{=} \int_{s=b}^{s=a} \frac{1}{\gamma(s)-z_0} \gamma'(s)\,ds = -\int_{s=a}^{s=b} \frac{\gamma'(s)\,ds}{\gamma(s)-z_0}$$

$$= -2\pi i \, \mathrm{Ind}_{\gamma}(z_0).$$

2) Let γ_1, γ_2 be closed contours and $a \in \mathbb{C} \setminus (\gamma_1^* \cup \gamma_2^*)$ and endpoint of γ_1 = inital point of γ_2. Then

$$\mathrm{Ind}_{\gamma_1 \cup \gamma_2}(a) = \mathrm{Ind}_{\gamma_1}(a) + \mathrm{Ind}_{\gamma_2}(a).$$

Pf Follows from linearity of path integrals.