

Thm 1.3 Let:  $G \stackrel{\text{open}}{\subseteq} \mathbb{C}$

$f = u + iv : G \rightarrow \mathbb{C}$  with continuous partials.

$f$  conformal on  $G$ .

Then  $f \in H(G)$  and  $f'(z) \neq 0 \quad \forall z \in G$ .

Pf. See Course Script.

Cor. Let  $f = u + iv : G \rightarrow \mathbb{C}$  have continuous partials.  $\& \quad G \stackrel{\text{open}}{\subseteq} \mathbb{C}$ .

$$[f \text{ is conformal}] \Leftrightarrow [f \in H(G) \ \& \ f'(z) \neq 0 \ \forall z \in G.]$$

Ch 2

§ 2.1 Contour/Path integrals

Def 2.1 A curve  $\gamma : [a, b] \rightarrow \mathbb{C}$  is rectifiable provided

$$l(\gamma) := \sup \left\{ \sum_{i=1}^n |\gamma(t_i) - \gamma(t_{i-1})| : a = t_0 < t_1 < \dots < t_n = b \right\} < \infty,$$

in which case,  $l(\gamma)$  is the length of  $\gamma$ . <draw yourself a picture>

Loosely speaking, if  $P_1 = \{t_0, \dots, t_n\} < P_2 := P_1 \oplus \text{more } t\text{'s}$ , then " $\sum$  for  $P_1$ "  $\leq$  " $\sum$  for  $P_2$ ".  
by intuition, what kind of  $\sum$  are we talking about? (Thm 2.3)

Def. Let  $\gamma : [a, b] \rightarrow \mathbb{C}$  be continuous.

<So  $\text{Re } \gamma, \text{Im } \gamma : [a, b] \rightarrow \mathbb{R}$  are continuous thus Riemann integrable.>

$$\text{Then } \int_a^b \gamma(t) dt := \int_a^b \text{Re } \gamma(t) dt + i \int_a^b \text{Im } \gamma(t) dt.$$

↑ these are Riemann integrals      ↑ these 2 are Riemann integrals

Prob.  $\gamma : [a, b] \rightarrow \mathbb{C}$  piecewise smooth  $\Rightarrow$



Rmk. If  $\gamma: [a, b] \rightarrow \mathbb{C}$  is piecewise smooth, then by the FTC for  $\mathbb{R}$ -valued fns,

$$\int_a^b \gamma'(t) dt = \gamma(b) - \gamma(a).$$

Lemma 2.2 If  $f: [a, b] \rightarrow \mathbb{C}$  is continuous, then

$$\left| \int_a^b f(t) dt \right| \leq \int_a^b |f(t)| dt$$

Sketch of Pf

$$0 \neq \left| \int_a^b f(t) dt \right| = \underbrace{e^{i\theta}}_{\substack{\exists \theta \in \mathbb{R} \\ \uparrow \\ \text{Re}}} \int_a^b f(t) dt = \int_a^b \text{Re}(e^{i\theta} f(t)) dt \leq \int_a^b |e^{i\theta} f(t)| dt = \int_a^b |f(t)| dt$$

$\text{Re } z \leq |z|$

Thm 2.3 Let  $\gamma: [a, b] \rightarrow \mathbb{C}$  be a piecewise smooth curve. Then

$$\langle \gamma \in C^1([a, b]) \rangle$$

(1)  $\gamma$  is rectifiable

(2)  $l(\gamma) = \int_a^b |\gamma'(t)| dt.$

Pf. LTGBG. WLOG,  $\gamma$  is smooth. Let  $M :=$

Pf. Let  $a = t_0 < \dots < t_n = b$  be a partition of  $[a, b]$ . Then

$$|\gamma(t_i) - \gamma(t_{i-1})| \stackrel{\text{FTC}}{=} \left| \int_{t_{i-1}}^{t_i} \gamma'(t) dt \right| \stackrel{\text{Lemma 2.2}}{\leq} \int_{t_{i-1}}^{t_i} |\gamma'(t)| dt$$

and so, where  $M := \sup_{t \in [a, b]} |\gamma'(t)| < \infty$  (since  $|\gamma'| \in C([a, b], \mathbb{R})$ )

$$\sum_{i=1}^n |\gamma(t_i) - \gamma(t_{i-1})| \leq \int_a^b |\gamma'(t)| dt \leq (b-a) \sup_{t \in [a, b]} |\gamma'(t)| \leq (b-a) M < \infty$$

So  $\gamma$  is rectifiable and

$$l(\gamma) \leq \int_a^b |\gamma'(t)| dt.$$

Towards showing (which is enough)

$$\int_a^b |\gamma'(t)| dt \leq l(\gamma), \quad (\text{WTS})$$

fix  $\epsilon > 0$ . Since  $\gamma'$  is uniformly continuous on  $[a, b]$ ,



• Since  $\gamma'$  is unif. cont. on  $[a, b]$ ,

$$\exists \delta > 0 \text{ st } |t-s| < \delta \Rightarrow |\gamma'(t) - \gamma'(s)| < \varepsilon. \quad (1)$$

• Find a partition  $a = t_0 < \dots < t_n = b$  w.s.t.  $\Delta t_i = t_i - t_{i-1} < \delta$

$$\left| \int_a^b |\gamma'(t)| dt - \sum_{i=1}^n |\gamma'(t_i)| \Delta t_i \right| < \varepsilon \quad (2)$$

$$\Delta t_i := t_i - t_{i-1} < \delta. \quad (3)$$

• For  $i = 1, \dots, n$ , estimate:

$$\left| |\gamma(t_i) - \gamma(t_{i-1})| - |\gamma'(t_i)| \Delta t_i \right|$$

$$\stackrel{\text{reverse}}{\leq} \left| \gamma(t_i) - \gamma(t_{i-1}) - \gamma'(t_i) \Delta t_i \right| \quad (4)$$

↓ FTC

$$\stackrel{\text{FTC}}{=} \left| \int_{t_{i-1}}^{t_i} \gamma'(t) dt - \int_{t_{i-1}}^{t_i} \gamma'(t_i) dt \right|$$

$$\stackrel{\text{Lemma 2.2}}{\leq} \int_{t_{i-1}}^{t_i} |\gamma'(t) - \gamma'(t_i)| dt \stackrel{(3) \& (1)}{<} \varepsilon \Delta t_i.$$

• Claim:  $\int_a^b |\gamma'(t)| dt \leq l(\gamma) + \varepsilon [b-a+1]$

$$\int_a^b |\gamma'(t)| dt - \varepsilon \stackrel{(2)}{\leq} \sum_{i=1}^n |\gamma'(t_i)| \Delta t_i$$

$$\stackrel{(4)}{\leq} \sum_{i=1}^n (|\gamma(t_i) - \gamma(t_{i-1})| + \varepsilon \Delta t_i)$$

$$\leq l(\gamma) + \varepsilon (b-a).$$

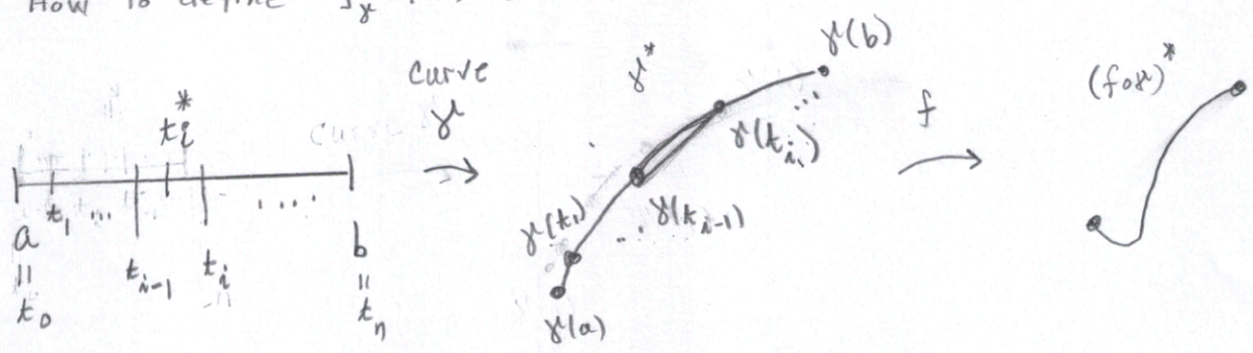
• Since  $\varepsilon > 0$  was arbitrary, WTS. holds.

□



Goal

How to define  $\int_{\gamma} f(z) dz$ ? Picture



Well...

$$\begin{aligned}
 \int_{\gamma} f(z) dz &= \sum_{i=1}^n \int_{\gamma|_{[t_{i-1}, t_i]}} f(z) dz \\
 &\approx \sum_{i=1}^n f(\gamma(t_i^*)) \int_{\gamma|_{[t_{i-1}, t_i]}} 1 dz \\
 &\approx \sum_{i=1}^n f(\gamma(t_i^*)) \underbrace{\text{"length of } \gamma|_{[t_{i-1}, t_i]}}_{\approx \vec{\gamma}(t_i) - \vec{\gamma}(t_{i-1})} \\
 &\approx \sum_{i=1}^n f(\gamma(t_i^*)) \frac{\gamma(t_i) - \gamma(t_{i-1})}{t_i - t_{i-1}} \cdot t_i - t_{i-1} \\
 &\approx \sum_{i=1}^n f(\gamma(t_i^*)) \gamma'(t_i^*) \Delta t_i \\
 &\approx \sum_{i=1}^n \int_{t_{i-1}}^{t_i} f(\gamma(t)) \gamma'(t) dt \\
 &= \int_a^b f(\gamma(t)) \gamma'(t) dt.
 \end{aligned}$$

Now to put the proper restrictions on  $f$  and  $\gamma$  so that "everything makes sense".



Def. Let:

- (1)  $\gamma: [a, b] \rightarrow \mathbb{C}$  be a piecewise smooth curve <sup>path/contour</sup>
- (2)  $f: \gamma^* \rightarrow \mathbb{C}$  be continuous.

Then the contour/path/line integral of  $f$  along  $\gamma$  is given by

$$\int_{\gamma} f(z) dz := \int_a^b f(\gamma(t)) \gamma'(t) dt.$$

Remark Just "pushing symbols" w/ "substitution"  $z = \gamma(t) \Rightarrow dz = \gamma'(t) dt$

Ex 2.4 Let  $\gamma: [a, b] \rightarrow \mathbb{C}$  be a piecewise smooth curve. Then

$$\int_{\gamma} 1 dz \stackrel{\text{def}}{=} \int_a^b 1 \cdot \gamma'(t) dt \stackrel{\text{FTC}}{=} \gamma(b) - \gamma(a).$$

Defs from old curves to new curves.

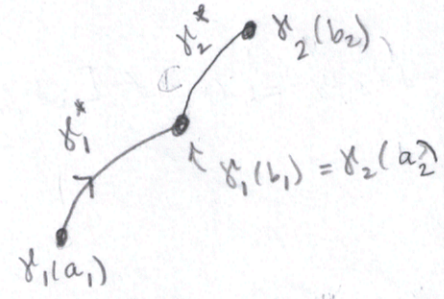
- (1) Let  $\gamma: [a, b] \rightarrow \mathbb{C}$  be a curve. It's a "reversely oriented" curve  $-\gamma: [a, b] \rightarrow \mathbb{C}$  is defined by  $-\gamma(t) := \gamma(a+b-t)$ .



- (2) Consider a pair of curves

$$\gamma_1: [a_1, b_1] \rightarrow \mathbb{C}$$

$$\gamma_2: [a_2, b_2] \rightarrow \mathbb{C}$$



s.t.  $\gamma_1(b_1) = \gamma_2(a_2)$ .

The join (or union)  $\gamma_1 \cup \gamma_2: [a_1, b_1 + b_2 - a_2] \rightarrow \mathbb{C}$  of  $\gamma_1$  and  $\gamma_2$  is given by

$$(\gamma_1 \cup \gamma_2)(t) = \begin{cases} \gamma_1(t) & a_1 \leq t \leq b_1 \\ \gamma_2(t + a_2 - b_1) & b_1 \leq t \leq b_1 + b_2 - a_2 \end{cases}$$

Note  $\gamma_1 \cup \gamma_2$  is again a curve, and

- $\gamma_1$  and  $\gamma_2$  smooth  $\nRightarrow \gamma_1 \cup \gamma_2$  smooth
- $\gamma_1$  and  $\gamma_2$  piecewise smooth  $\Rightarrow \gamma_1 \cup \gamma_2$  piecewise smooth.



Prop 2.5 Let:

- $\gamma, \gamma_1, \gamma_2: [a, b] \rightarrow \mathbb{C}$  be a piecewise smooth curve
- $f, g: \gamma^* \rightarrow \mathbb{C}$  be a continuous function.
- $\alpha, \beta \in \mathbb{C}$

Then the following hold.

$$(1) \int_{-\gamma} f(z) dz = - \int_{\gamma} f(z) dz$$

$$(2) \int_{\gamma_1 \cup \gamma_2} f(z) dz = \int_{\gamma_1} f(z) dz + \int_{\gamma_2} f(z) dz.$$

$$(3) \left| \int_{\gamma} f(z) dz \right| \leq \left[ \max_{z \in \gamma^*} |f(z)| \right] \cdot [l(\gamma)]$$

"ML" →

Question: What is fundamentally wrong with  $\left| \int_{\gamma} f(z) dz \right| \leq \int_{\gamma} |f(z)| dz$ ?  
Hint: Consider  $\gamma(t) = it$  and  $f(z) \equiv 1$  so  $\int_{\gamma} |f(z)| dt = \int_0^1 |1| | \gamma'(t) | dt = i$ .

(4) Independence of parametrization

$$\gamma \circ \tau: [a_1, b_1] \xrightarrow[\tau' > 0]{\tau} [a, b] \xrightarrow{\gamma} \mathbb{C}$$

an orientation preserving reparametrization of  $\gamma$

Let  $\tau: [a_1, b_1] \rightarrow [a, b]$  be a smooth onto function with  $\tau' > 0$ . Then

$$\int_{\gamma \circ \tau} f(z) dz = \int_{\gamma} f(z) dz.$$

$$(5) \int_{\gamma} [\alpha f(z) + \beta g(z)] dz = \alpha \int_{\gamma} f(z) dz + \beta \int_{\gamma} g(z) dz.$$

pf For details, see script. Here's the ideas.

- (1)  $(\gamma \circ \tau)(t) \stackrel{\text{def}}{=} \gamma(a+b-t) \Rightarrow (-\gamma)'(t) = -\gamma'(a+b-t)$ . Then just use def. of path integral
- (2) Follows from defs of path integral and  $\gamma_1 \cup \gamma_2$ . <join>
- (3)  $\left| \int_{\gamma} f(z) dz \right| = \left| \int_a^b f(\gamma(t)) \gamma'(t) dt \right| \leq \int_a^b |f(\gamma(t))| |\gamma'(t)| dt \leq M \int_a^b |\gamma'(t)| dt = M l(\gamma)$   
↑ Lemma 2.2 ↑ by R-variable calc. ↑ Thm 2.3

(4) Follows from chain rule ⊕ change of variable rule for R-variable calc.

(5) Follows from the defs ⊕ R-variable calc.