Ch 2 Integration over Contours

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§ 1.1 Curves and Contours : The Basics

Defs A curve is a continuous map $\gamma : [a,b] \to \mathbb{C}$ ($[a,b] \subseteq \mathbb{R}$) (so $\gamma \in C([a,b])$)

- γ^* denotes $\gamma([a,b]) \overset{\text{note}}{\subseteq} \mathbb{C}$
- $[a,b]$ = parameter interval of γ
- $\gamma(a)$ = initial point of γ
- $\gamma(b)$ = end point of γ
- γ is closed $\iff$ $\gamma(a) = \gamma(b)$
- γ is smooth $\iff$ γ' exists and is continuous on $[a,b]$
  $\overset{\text{notation}}{\iff}$ $\gamma \in C^1([a,b])$

Example What can you say about the 2 curves

$$\gamma_1 : [0, 2\pi] \to \mathbb{C}, \quad \gamma_1(\theta) = e^{i\theta}$$

$$\gamma_2 : [0, 4\pi] \to \mathbb{C}, \quad \gamma_2(\theta) = e^{-i\theta} \ ?$$

Well, for one thing, $\gamma_1^* = \gamma_2^* = \{ z \in \mathbb{C} \mid |z| = 1 \}$.

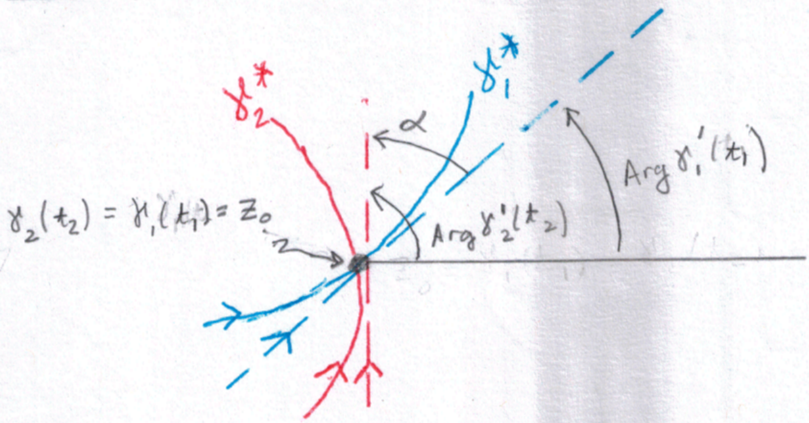
Defs A path (or contour) $\gamma : [a,b] \to \mathbb{C}$ is a piecewise smooth curve. (bs)

- γ is closed $\iff$ $\gamma(a) = \gamma(b)$.
- γ is simple $\iff$ [if $a \le s < t \le b$ and $\gamma(s) = \gamma(t)$ then $s = a$ and $t = b$]

  $\iff$ "loosely speaking, γ^* doesn't cross itself except perhaps at the endpts"

Ex 1.1 From Course Script, p 13. Read.

Def via a picture. The angle α at $z_0 \in \mathbb{C}$ between smooth curves γ_1 and γ_2 that intercept at z_0 (from γ_1 to γ_2) is "α":



Rmk: We also assume that t_i is in the interior of the parameter interval of γ_i.

Note ① α has a direction (cw/ccw) as well as a magnitude.

② If $\gamma_1'(t_1) \neq 0$ and $\gamma_2'(t_2) \neq 0$, then:

$$\alpha \in \arg \gamma_2'(t_2) - \arg \gamma_1'(t_1)$$

$$-\pi < \alpha \leq \pi.$$

Def. Consider a mapping $f: G \to \mathbb{C}$ where $G \overset{\text{open}}{\subseteq} \mathbb{C}$.

① f preserves angles at $z_0 \in G$ $\overset{\text{def}}{\Longleftrightarrow}$

[for each pair of smooth curves γ_1 and γ_2 in G that intercept at z_0 s.t. if $\gamma_1(t_1) = z_0 = \gamma_2(t_2)$ then $\gamma_i'(t_i) \neq 0$ for each i,
the angle btw. γ_1 and γ_2 at z_0 =
the angle btw $f \circ \gamma_1$ and $f \circ \gamma_2$ at $f(z_0)$.]

② f is conformal (on G) $\overset{\text{def}}{\Longleftrightarrow}$

f preserves angles at each $z_0 \in G$.

Thm 1.2 Let: $G \overset{\text{open}}{\subset} \mathbb{C}$

$f \in H(G)$

$f'(z) \neq 0 \ \forall z \in G.$

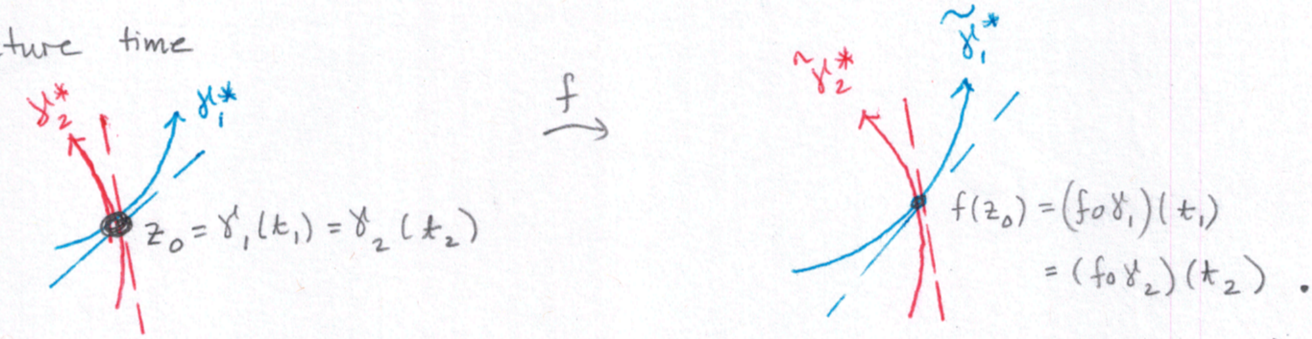
Then f is conformal on G.

Pf. LTGBG. Fix $z_0 \in G$. For $i \in \{1,2\}$, consider smooth curves

$$\gamma_i : [a_i, b_i] \longrightarrow G$$

s.t., for some $t_i \in (a_i, b_i)$, $\gamma_i(t_i) = z_0$ and $\gamma_i'(t_i) \neq 0$.

< Picture time



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Define

$$\tilde{\gamma}_i := f \circ \gamma_i : [a_i, b_i] \to \mathbb{C}.$$

By the chain rule,

$$\boxed{\tilde{\gamma}_i'(t_i) = f'(z_0) \cdot \gamma_i'(t_i)} \quad . \qquad (1)$$

Note $\tilde{\gamma}_i'(t_i) \neq 0$ b/c by assumptions, $f'(z_0) \neq 0$ & $\gamma_i'(t_i) \neq 0$.

Thus, by (1), <as sets>

$$\arg \tilde{\gamma}_2'(t_2) = \arg f'(z_0) + \arg \gamma_2'(t_2)$$

$$\arg \tilde{\gamma}_1'(t_1) = \arg f'(z_0) + \arg \gamma_1'(t_1)$$

$$\arg \tilde{\gamma}_2'(t_2) - \arg \tilde{\gamma}_1'(t_1) = \arg \gamma_2'(t_2) - \arg \gamma_1'(t_1).$$

∎

Thm 1.3 Let: $G \overset{open}{\subseteq} \mathbb{C}$

$f = u + iv : G \to \mathbb{C}$ with continuous partials.

f conformal on G.

Then $f \in H(G)$ and $f'(z) \neq 0 \quad \forall z \in G$.

Pf. See Course Script.

Cor. Let $f = u + iv : G \to \mathbb{C}$ have continuous partials. & $G \overset{open}{\subseteq} \mathbb{C}$.

$$[f \text{ is conformal}] \iff [f \in H(G) \text{ \& } f'(z) \neq 0 \ \forall z \in G.]$$

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§ 2. Contour / Path integrals

Def 2.1 A curve $\gamma : [a,b] \to \mathbb{C}$ is rectifiable provided

$$\ell(\gamma) := \sup \left\{ \sum_{i=1}^{n} |\gamma(t_i) - \gamma(t_{i-1})| : a = t_0 < t_1 < \dots < t_n = b \right\} < \infty,$$

in which case, $\ell(\gamma)$ is the length of γ. <draw yourself a picture>

Loosely speaking, if $P_1 := \{t_0, \dots, t_n\} \prec P_2 := P_1 \oplus$ more t's, then "Σ for P_1" $\overset{\Delta}{\leq}$ "Σ for P_2".

Δ(Thm 2.3)

Def. Let $\gamma : [a,b] \to \mathbb{C}$ be continuous.

<So $\operatorname{Re}\gamma, \operatorname{Im}\gamma : [a,b] \to \mathbb{R}$ are continuous thus Riemann intergable.>

Then $$\int_a^b \gamma(t)\,dt := \int_a^b \operatorname{Re}\gamma(t)\,dt + i \int_a^b \operatorname{Im}\gamma(t)\,dt.$$

[these 2 are Riemann integrals]